

NON-LINEAR MATRIX EQUATIONS: EQUILIBRIUM ANALYSIS OF MARKOV CHAINS

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1. Introduction:

In the research area of one dimensional stochastic processes, Markov chains acquire special importance due to large number of applications. One of the simplest possible continuous time Markov chains, namely birth-and-death process arises naturally in many queueing models. Such a Markov chain has an efficient recursive solution for the equilibrium probabilities. Specifically the equilibrium probabilities form a geometric sequence. The common ratio/recursion constant is the solution of a quadratic equation.

Evans and Wallace considered a stochastic process called the Quasi-Birth-and-Death (QBD) process as a natural generalization of the birth-and-death process. They showed that for a block-Jacobi generator of continuous parameter Markov processes (called QBD processes), the stationary probability vector \bar{X} may be partitioned into $1 \times n$ vectors \bar{x}_k , $k \geq 0$ which are given by

$$\bar{x}_k = \bar{x}_0 R^k \text{ for } k \geq 0 \quad \dots\dots\dots(1.1)$$

where the square matrix R is the minimal non-negative solution of a matrix-quadratic equation. An equilibrium probability vector \bar{X} , which satisfies equation (1.1) will be called a *matrix-geometric probability vector*.

A natural question which arises is whether the matrix geometric recursive solution exists for the equilibrium probabilities of a more general class of Markov processes. Marcel Neuts has shown [Neu1] that a matrix geometric recursive solution exists for the equilibrium probabilities of a large class of processes, called G/M/1-type Markov processes. This is fortunate since these processes provide good stochastic models for various problems arising in queueing and inventory theories.

The state space, E of a G/M/1-type Markov process has the following form:

$$E = \{ (i, j) : i \geq 0, 1 \leq j \leq n \} \quad \dots\dots\dots(1.2)$$

in which "n" is finite but otherwise arbitrary. This state space can be clearly decomposed into levels by performing a lexicographic partitioning on the first state variable. For each level k, an equilibrium probability vector $\bar{\pi}_k$ can then be defined. It is given by

$$\bar{\pi}_k = [\pi_{k,1} \ \pi_{k,2} \ \dots\dots\pi_{k,n}] \quad \dots\dots\dots(1.3)$$

where $\pi_{k,j}$ is the equilibrium probability that the process is in the state (k,j). The entire set of limiting probabilities for the process is then specified by the infinite-dimensional vector

$$\bar{\pi} = [\bar{\pi}_0 \ \bar{\pi}_1 \ \bar{\pi}_2 \ \dots \ \bar{\pi}_i \ \dots]. \quad \dots\dots\dots(1.4)$$

In the case of a G/M/1-type Markov process, the transitions between states in various levels are restricted. Specifically, upward transitions from level k can only reach level (k+1); downward transitions from level k can reach, in one transition, any level j for j < k. Thus, with the above partitioning and ordering of the state space into levels, the generator matrix **Q** of a G/M/1-type Markov process X_t has the following form:

$$Q = \begin{bmatrix} B_0 & C_0 & 0 & 0 & 0 & 0 & \dots \\ B_1 & A_1 & A_0 & 0 & 0 & 0 & \dots \\ B_2 & A_2 & A_1 & A_0 & 0 & 0 & \dots \\ B_3 & A_3 & A_2 & A_1 & A_0 & 0 & \dots \\ B_4 & A_4 & A_3 & A_2 & A_1 & A_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad \dots\dots\dots(1.5)$$

in which all matrices, except possibly some of the boundary matrices, are n x n.

The equilibrium probability vector $\bar{\pi}$ is the unique solution to

$$\bar{\pi} \mathbf{Q} = \mathbf{0} \quad \dots\dots\dots(1.6)$$

Solving the equation (1.6) for $\bar{\pi}$ directly is time consuming and very cumbersome. It is desirable to have a recursive method for computing $\bar{\pi}$. In other words, we would like to find an expression for $\bar{\pi}_k$ in terms of $\bar{\pi}_j$ for j < k.

Neuts showed [Neu1, Neu2] that $\bar{\pi}_k$ can be found in terms of $\bar{\pi}_{k-1}$ as follows

$$\bar{\pi}_k = \bar{\pi}_{k-1} \mathbf{R}, \quad \dots\dots\dots(1.7)$$

Where the matrix **R** is the minimal nonnegative solution of

$$\sum_{i=0}^{\infty} X^i A_i = 0 \quad \dots\dots\dots(1.8)$$

The A_i matrices in (1.8) are the submatrices of the generator **Q** in (1.5). Thus

A matrix geometric recursion exists for the equilibrium probabilities in $\bar{\pi}$.

The rate matrix **R** can be utilized to express most of the steady state characteristics of the G/M/1-type Markov processes. Thus, efficient algorithms for the computation of this matrix are therefore of considerable interest in applied probability.

Traditionally, computation of rate matrix is approached utilizing iterative procedures. Since the direct iteration scheme requires large processing time, a Picard iteration scheme and other improved schemes have been proposed which substantially decrease the number of iterations [Neu2]. Based on an experimental study of various schemes [Rama1], a new more efficient algorithm is proposed in [Rama2]. These iterative procedures for the computation of rate matrix truncate the power series

in (1.8), thus effectively making a polynomial approximation to it. Thus these iterative procedures are approximate, time consuming in nature. Also convergence problems could occur. In [LaR], a logarithmic reduction algorithm is proposed for QBD processes.

In [RKC], an alternative procedure to compute the eigenvalues, eigenvectors of R is proposed. Thus, the Jordan canonical form representation (Closed form Representation) of R is found. Even in this work [RKC], the matrix power series in (1.8) is truncated and the solution of a matrix polynomial equation is determined. But in this paper we consider an irreducible, positive recurrent G/M/1-type Markov process, whose rate matrix satisfies a matrix power series equation (and not a matrix polynomial equation). Method for computation of Jordan Canonical Form representation of rate matrix is proposed.

This research paper is organized as follows. In section 2, the general problem of solving matrix power series equation Of the form in (1.8) is considered. Some interesting lemmas and Theorems are proved. By exploiting the properties of A_i matrices (sub-matrices in the generator), Theorems related to the computation of eigenvalues and left eigenvectors are proved in Section 3. Thus method of computation of rate matrix is discussed in Section 4. It is briefly discussed how the generalized eigenvectors and thus the Jordan canonical form of rate matrix can be computed. In Section 4, by using a simple transformation, it is described how the results developed for computation of \mathbf{R} hold *mutatis mutandis* for the computation of matrix geometric recursion matrix \bar{R} arising in the equilibrium analysis of discrete time Markov chains of G/M/1-type. Furthermore, it is briefly described how a similar method can be utilized in the equilibrium analysis of M/G/1-type Markov chains.

2.1 Matrix Power Series Equations: Solution Techniques: Algebraic Geometry:

Consider a matrix power series equation of the following form:

$$\sum_{i=0}^{\infty} X^i D_i = 0, \quad \dots\dots\dots(2.1)$$

where the coefficient matrices $\{ D_i ; i \geq 0 \}$ are known and \mathbf{X} is the unknown matrix. For the sake of convenience, consider the case where the elements of D_i 's are from the field of real/complex numbers. The following Lemma provides a method of determining the eigenvalues of all possible solutions of (2.1).

Lemma 1: Consider a matrix \mathbf{Y} which satisfies (2.1) and let

$$H(\lambda) = \sum_{j=0}^{\infty} \lambda^j D_j. \quad \dots\dots\dots(2.2)$$

Then $H(\lambda)$ has the following representation

$$H(\lambda) = (\lambda I - Y) \left(\sum_{j=0}^{\infty} \lambda^j N_j \right) \quad \dots\dots\dots(2.3)$$

Where
$$N_j = \sum_{k=j+1}^{\infty} Y^{k-j-1} D_k \dots\dots\dots(2.4).$$

Thus the characteristic polynomial of every solution of (2.1) divides the determinantal transcendental function

$$\text{Det} (H(\lambda)) = \text{Det} (\sum_{j=0}^{\infty} \lambda^j D_j). \dots\dots\dots(2.5)$$

Proof: It is easy to see that if λ is an eigenvalue of solution Y , then it is a root of the determinantal transcendental function $\text{Det} (\mathbf{G}(\lambda))$. The more challenging part is the necessity part.

The Lemma's claim deals with factorization of matrix valued analytic functions. An important research paper on this topic was written by M.L.J. Hautus [Hau] entitled, "Operator Substitution". Specifically, the above Lemma follows from claim 2 in property 2.6 (Remainder Theorem).

From the point of view of symbolic algebra, the following derivation provides the required factorization. The approach originated in a discussion with Prof. Balasubramanian.

$$\begin{aligned} \sum_{j=0}^{\infty} \lambda^j D_j &= \sum_{j \geq 1} (\lambda^j I - Y^j) D_j \\ &= \sum_{j=1}^{\infty} (\lambda I - Y) (\lambda^{j-1} I + \lambda^{j-2} Y + \dots + Y^{j-1}) D_j \\ &= (\lambda I - Y) \left(\sum_{j=0}^{\infty} \lambda^j N_j \right), \text{ where } \dots\dots\dots(2.6) \end{aligned}$$

$$N_j = \sum_{k=j+1}^{\infty} Y^{k-j-1} D_k \dots\dots(2.7) \quad \mathbf{QED}$$

The following Theorem is a generalization of a result in [Gan] for matrix polynomial equations.

Theorem 1: Consider a matrix power series equation of the form in (2.1) i.e.

$$\sum_{i=0}^{\infty} X^i D_i = 0$$

Let the dimension of \mathbf{X} be 'n'. Let there be "finitely" many, say 'm' ($m > n$) roots of the transcendental function $\text{Det} (\sum_{j=0}^{\infty} \lambda^j D_j)$. Then all possible solutions of (2.1) are

divided into atmost $\binom{m}{n}$ equivalence classes and solution in each class is determined

As the solution of a linear system of equations.

Proof: From equation (2.3); we know that the characteristic polynomial of every solution divides the transcendental function $\text{Det} \left(\sum_{j=0}^{\infty} \lambda^j D_j \right)$. This inference follows from the following equation

$$\text{Det}(H(\lambda)) = \text{Det}(\lambda I - Y) \text{Det} \left(\sum_{j=0}^{\infty} \lambda^j N_j \right) \dots \dots \dots (2.8)$$

Since there are at most “m” roots of $\text{Det} \left(\sum_{j=0}^{\infty} \lambda^j D_j \right)$ and since each solution of (2.1) has “n” roots, the solutions are divided into $\binom{m}{n}$ classes. It is easy to see that the set of solutions in each class constitute an equivalence class.

Let the Jordan canonical form of a solution in “ith” class be

$$T E T^{-1}.$$

It should be noted that the set of eigenvalues determine D. Substituting, the solution in (2.1), we have

$$\sum_{i=0}^{\infty} T E^i T^{-1} D_i = 0$$

The above equation is equivalent to the following one.

$$T \left[\sum_{i=0}^{\infty} E^i T^{-1} D_i \right] = 0.$$

Since T is non-singular, we necessarily have that

$$\left[\sum_{i=0}^{\infty} E^i T^{-1} D_i \right] = 0. \dots \dots \dots (2.9)$$

Since eigenvalues are known, E is known. Thus we have to solve for T using the above equation. It is clear that the above system of equations are linear. **QED.**

Lemma 2: Rank (Y) ≥ Rank (D₀).

Proof: Let \bar{f} be a vector in the left null space of Y.

$$\bar{f} \left(\sum_{i=0}^{\infty} Y^i D_i \right) = 0,$$

implies $\bar{f} D_0 = 0$. Hence \bar{f} lies in the left null space of D₀. Thus,

$$\text{Rank} (D_0) \leq \text{Rank} (Y). \dots (2.10) \quad \mathbf{QED}$$

Remark:

From the above Lemma, it is clear that if D₀ is nonsingular, then every solution of (2.1) is nonsingular.

A. Reduction of a Matrix Power Series Equation to a Matrix Polynomial Equation:

Consider a particular solution Y of (2.1). In the following Lemma it is shown that if the eigenvalues of Y are known, then the matrix power series equation in (2.1) can be reduced to a matrix polynomial equation of the following form

$$\sum_{i=0}^l Y^i G_i = 0, \text{ where}$$

G_i 's are matrix series in G_i .

Lemma 3: If Y satisfies (2.1) with the matrix power series being absolutely convergent and the characteristic polynomial of Y is known, then

$$\sum_{j=0}^{\infty} Y^j D_j = \sum_{j=0}^{n-1} Y^j \overline{D}_j, \quad \dots\dots\dots(2.11)$$

where the matrices \overline{D}_j 's are obtained from the D_j 's and the coefficients of the characteristic polynomial of Y .

Proof: Let the characteristic polynomial of Y be

$$\alpha_0 + \alpha_1 \lambda + \dots + \alpha_n \lambda^n.$$

By the Cayley-Hamilton Theorem,

$$\alpha_0 I + \alpha_1 Y + \dots + \alpha_n Y^n = 0.$$

Equivalently,

$$Y^n = -\frac{\alpha_0}{\alpha_n} I - \frac{\alpha_1}{\alpha_n} Y - \dots - \frac{\alpha_{n-1}}{\alpha_n} Y^{n-1}.$$

Multiplying the above equation by Y and substituting for Y^n , an expression for Y^{n+1} can be obtained in terms of $I, Y, Y^2, \dots, Y^{n-1}$.

$$Y^{n+1} = \beta_0^{(n+1)} I + \beta_1^{(n+1)} Y + \dots + \beta_{n-1}^{(n+1)} Y^{n-1},$$

where the coefficients $\beta_i^{(n+1)}$'s are related to the α_i 's. The process can be repeated to find Y^m for arbitrary m in terms of $I, Y, Y^2, \dots, Y^{n-1}$.

Thus

$$Y^m = \sum_{j=0}^{n-1} \beta_j^{(m)} Y^j \quad \dots\dots\dots(2.12)$$

Since the matrix power series on the left hand side of (2.1) is absolutely convergent, rearranging the terms will not affect the sum. Hence, substituting the above expression for Y^m in (2.1) and regrouping the terms yields the following equation

$$\sum_{j=0}^{\infty} Y^j D_j = \sum_{j=0}^{n-1} Y^j \overline{D}_j \quad \text{QED}$$

To illustrate the utility of above results, a special class of nonlinear matrix equations which arise in the equilibrium analysis of G/M/1-type Markov processes are considered. In the following discussion, we consider G/M/1-type

Markov processes which are irreducible and positive recurrent.

2.2 System of Non-Linear Matrix Equations: Algebraic Geometry:

Now let us consider non-linear matrix equations which are of the following form:

- Matrix Polynomial equations i.e. the unknown as well as coefficient matrices are compatible matrices e.g

$$X^m A_m + X^{m-1} A_{m-1} + \dots + X A_1 + A_0 = 0 \dots\dots\dots(2.13)$$

In the above equation X as well as A_i 's are compatible matrices.

- Matrix Power Series Equations:

$$\sum_{i=0}^{\infty} X^i A_i = 0 \dots\dots\dots(2.14)$$

i.e unknown X as well as coefficients A_i 's are matrices.

A. MATRIX POLYNOMIAL EQUATIONS:

It is well known that one of the central goals of algebraic geometry is to determine the existence, uniqueness as well as cardinality of solutions of a system of multi-variate polynomial equations. Bezout's Theorem is an important result in this direction. **It is now shown with an example that the solution of (2.13) falls in the domain of algebraic geometry.**

Let us consider a special case of (2.13) i.e. a matrix quadratic equation [RaC]

$$X^2 A_2 + X A_1 + A_0 = 0, \dots\dots\dots(2.15)$$

where $\{X, A_0, A_1, A_2\}$ are matrices. For the sake of illustration, let us consider the simple case where these matrices are 2×2 matrices. Let the unknowns in the matrix X be denoted as $\{x_1, x_2, x_3, x_4\}$. Also, let $\{a_{ij}^{(k)}; 1 \leq i, j \leq 2\}$ be the entries of A_k for $k=0,1,2$. It is easy to see that the equation (2.15) represents a system of 4 equations in unknowns $\{x_1, x_2, x_3, x_4\}$ of highest degree 2. The four equations are given by

$$\begin{aligned} (x_1^2 + x_2 x_3) a_{11}^{(2)} + (x_1 x_2 + x_2 x_4) a_{21}^{(2)} + (x_1) a_{11}^{(1)} + (x_2) a_{21}^{(1)} + a_{11}^{(0)} &= 0 \\ (x_1^2 + x_2 x_3) a_{12}^{(2)} + (x_1 x_2 + x_2 x_4) a_{22}^{(2)} + (x_1) a_{12}^{(1)} + (x_2) a_{22}^{(1)} + a_{12}^{(0)} &= 0 \dots\dots\dots(2.16) \\ (x_3 x_1 + x_4 x_3) a_{11}^{(2)} + (x_3 x_2 + x_4^2) a_{21}^{(2)} + (x_3) a_{11}^{(1)} + (x_4) a_{21}^{(1)} + a_{21}^{(0)} &= 0 \\ (x_3 x_1 + x_4 x_3) a_{12}^{(2)} + (x_3 x_2 + x_4^2) a_{22}^{(2)} + (x_3) a_{12}^{(1)} + (x_4) a_{22}^{(1)} + a_{22}^{(0)} &= 0 \end{aligned}$$

By a natural generalization to matrix polynomial equations, it is clear that we have a STRUCTURED system of multi-variate polynomial equations. The solution of an ARBITRARY matrix polynomial equation is discussed in [Gan]. The techniques of LINEAR ALGEBRA are applicable to this CLASS OF NON-LINEAR matrix equations. Thus some STRUCTURED problems in ALGEBRAIC GEOMETRY can be solved using LINEAR ALGEBRA techniques.

CONJECTURE: It is conjectured that ARBITRARY multi-variate polynomial equations can be imbedded in PROPERLY CHOSEN TENSOR VARIATE POLYNOMIAL EQUATIONS.

MULTI-VARIATE POWER SERIES EQUATIONS:

As in the case of matrix polynomial equations, it is easy to see that matrix power series equations represent a STRUCTURED class of multi-variate power series equations. Thus, it is shown in this research paper that such class of multi-variate equations can be solved using the techniques of linear algebra.

CONJECTURE: It is conjectured that ARBITRARY multi-variate power series equations can be IMBEDDED in a TENSOR VARIATE POWER SERIES EQUATIONS.

3 Computation of Eigenvalues and Eigenvectors of Rate Matrix:

The rate matrix \mathbf{R} is the minimal nonnegative solution of the matrix power series equation

$$\sum_{i=0}^{\infty} R^i A_i = 0 \quad \dots\dots(3.1)$$

Where the A_i 's are the submatrices in the generator matrix \mathbf{Q} . On invoking the Lemma 1, we realize that the eigenvalues of \mathbf{R} are a subset of the roots of the determinantal transcendental function

$$K(\lambda) = \text{Det} \left(\sum_{j=0}^{\infty} \lambda^j A_j \right) = \text{Det} (\mathbf{G}(\lambda)) \quad \dots\dots\dots(3.2)$$

In general $K(\lambda)$ is a transcendental function. The spectral radius of \mathbf{R} , $\text{Sp}(\mathbf{R})$, is strictly less than one, since the irreducible Markov process X_t is positive recurrent. Hence, "n" of the roots of $K(\lambda)$ lie strictly inside the unit circle in the complex plane. But in order to determine the eigenvalues of \mathbf{R} unambiguously from the roots of $K(\lambda)$, it still remains to localize the other roots to a distinct region of the complex plane. Theorem (2) in this section will provide the desired result.

Remark: Invoking results in sub-section (A) (in section 2), the matrix power series equation in (3.1) can be reduced to a matrix polynomial equation. The coefficient matrices in this "reduced" equation are obtained from those in equation (3.1) and the coefficients of characteristic polynomial of \mathbf{R}

If the coefficient matrices in the resulting, "reduced" matrix polynomial equation have the same properties as those in the equation in [RKC], we can directly invoke the results in research paper [RKC] to arrive at more general conclusions (method for computation of eigenvalues and eigenvectors of \mathbf{R}). But **it is not clear how to** deduce that the coefficient matrices of the "reduced" equation have the same properties as those of equation in section 3 of [RKC]. Thus we are naturally led to the following discussion.

Relationship to the Method based on Complex Analysis:

The results in this paper relating to equilibrium analysis of G/M/1-type Markov chains are purely based on algebraic arguments. In the following, these results are related to those obtained using the complex analysis method.

In the complex analysis method, one starts out to construct an equilibrium probability vector $\bar{\pi} = [\bar{\pi}_1 \bar{\pi}_2 \dots]$ of the form

$$\bar{\pi}_i = \sum_{j=1}^n \bar{a}_j \eta_j^i, \quad \dots\dots\dots(3.3)$$

where the (usually complex) numbers $\eta_j, 1 \leq j \leq n$ lie strictly inside the unit circle.

These and the row vectors \bar{a}_j are to be determined. η_k 's are normally determined from the zeros of

$$\varphi(\lambda) = \text{Det}(\lambda I - A^*(\lambda)) \quad \text{where} \quad A^*(\lambda) = \sum_{j=0}^{\infty} A_j \lambda^j, \quad \text{for } |\lambda| \leq 1, \dots\dots\dots(3.4)$$

It should be noted that $\varphi(\lambda)$ is usually a transcendental function. By an application of Rouché's Theorem and the perturbation theory for matrices, it is then shown that $\varphi(\lambda)$ has exactly "n" zeros $\eta_j, 1 \leq j \leq n$ satisfying $|\eta_j| \leq 1$. Detailed arguments must then be employed to find conditions under which all the zeros $\eta_j, 1 \leq j \leq n$ are strictly within the unit circle. These conditions, found for specific models, are in essence, special cases of the necessary and sufficient conditions for the positive recurrence of the Markov chain derived in [Neu1]. Once the zeros η_k are determined, their multiplicities are examined to find if the trial solution in (3.2) needs to be modified in case multiple zeros are present. For very special cases such an examination is carried out in [Bai], [BGK] and [Cin1].

It is evident that when the multiplicity of the zeroes of $\varphi(\lambda)$ is one or more, the matrix geometric recursion for the equilibrium probability vector is equivalent to the form in (3.2) and its modification. It is immediate to see if η_j, \bar{a}_j are the eigenvalue and left eigenvector of \mathbf{R} respectively, then

$$\bar{a}_j A^*(\eta_j) = \mathbf{0}.$$

Lemma 1 shows that the above condition is also sufficient. Now on utilizing the condition that the Markov chain is positive recurrent if and only if $\text{SP}(\mathbf{R}) < 1$ and the Localization Theorem proven in the sequel, the eigenvalues of \mathbf{R} can be determined by computing the zeros of the transcendental function $\text{Det}(\mathbf{K}(\lambda))$ which are within the unit circle. Thus the Rouché's Theorem is not invoked to infer that there are "n" zeroes of $\varphi(\lambda)$ which are on or inside the unit circle. Furthermore, no diagonalizability constraint on \mathbf{R} is imposed which is normally employed in complex analysis based methods. A characterization of the generalized left eigenvectors of the rate matrix can be derived along the lines of Lemma 5 in [RKC].

In the following Lemma, a new proof of equality (1.7.12) in [Neu1, p.33], is provided based on the results in section 2 on matrix power series equations. This Lemma will be utilized in the proof of Localization Theorem.

In [RKC], several results that are utilized in the proof of Localization

Theorem are derived. The matrix power series version of the results can easily be derived through a simple generalization of the arguments. Details are avoided here for brevity. Only important results are stated here without detailed proof.

Utilizing the definition of $K(\lambda)$ and Lemma (2.1), we have

$$K(\lambda) = \text{Det}(\lambda I - R) \text{Det}(N(\lambda)) \dots\dots\dots(3.5)$$

Theorem 2: $N(\lambda)$ is a diagonally dominant matrix if the absolute value of λ is strictly less than one.

Proof: The Theorem follows from a proof argument very similar to the one utilized in [RKC]. The detailed argument is avoided for brevity. **QED.**

The following Theorem is the main result utilized to determine the eigenvalues of R from the zeroes of the determinantal transcendental function $K(\lambda)$.

Theorem 3 (Localization Theorem): Given an irreducible, positive recurrent Markov process, suppose that $\sum_{j=1}^{\infty} A_j$ is irreducible. Then λ is an eigenvalue of the rate

matrix R if and only if $K(\lambda) = \text{Det}(\mathbf{G}(\lambda)) = 0$ and $|\lambda| < 1$, where $\mathbf{G}(\lambda) = \sum_{j=0}^{\infty} \lambda^j A_j$.

Proof: There are two approaches to proving the Theorem. The first approach utilizes the same argument as that utilized in complex analysis based method. Specifically Rouches Theorem is invoked as in [Tak], [Bai], [BGK] and [Cin1].

The second approach is a generalization of the proof argument utilized in [RKC]. This approach utilizes Theorem 2 stated above. Details are avoided for brevity. **QED.**

The above Localization Theorem provides a method of finding the eigenvalues of the rate matrix. In the following Lemma it is shown that the above Theorem also provides a method of computing the left eigenvectors of the rate matrix.

Lemma 3: The row vector \bar{u} is a left eigenvector corresponding to an eigenvalue λ of R if and only if

$$\bar{u} \left(\sum_{j=0}^{\infty} \lambda^j A_j \right) = \bar{u}(\mathbf{G}(\lambda)) = \mathbf{0}, \quad \dots\dots\dots(3.6)$$

Proof: Suppose λ is an eigenvalue corresponding to the left eigenvector \bar{u} . Necessary part of the above assertion follows from the definition of left eigenvector and from (3.1). Since R satisfies (3.1) and (3.1) is of the same form as (2.1), Lemma 1 can be applied. Hence

$$\sum_{j=0}^{\infty} \lambda^j A_j = (\lambda I - R) N(\lambda).$$

Now suppose that \bar{u} satisfies (). On using ()

$$\bar{u} (\lambda I - R) N(\lambda) = 0.$$

But the spectral radius of R is strictly less than one and so the absolute value of

λ is strictly less than one. By the Localization Theorem, $N(\lambda)$ is non-singular if the absolute value of λ is strictly less than one. Therefore we have that

$$\bar{u} (\lambda I - R) = 0.$$

Hence \bar{u} is a left eigenvector of \mathbf{R} .

QED.

In [RKC, Lemma 5], an alternative characterization of left eigenvectors of rate matrix \mathbf{R} is provided. It can be generalized to the matrix power series equation case. Thus the generalized left eigenvectors can be determined.

4 Computation of Rate Matrix of a G/M/1-Type Markov Process:

Consider the case when the rate matrix is diagonalizable. One sufficient case for diagonalizability is that all the eigenvalues of \mathbf{R} are distinct. When the rate matrix is diagonalizable, it can be computed through the spectral representation approach.

The spectral representation of \mathbf{R} is of the following form.

$$\mathbf{R} = T C T^{-1} \dots\dots\dots(4.1)$$

where C is a diagonal matrix with the p non-zero eigenvalues of \mathbf{R} on the diagonal and the rows of the matrix T^{-1} are the left eigenvectors of \mathbf{R} . The eigenvalues of \mathbf{R} can be found from the roots of the determinantal transcendental function $K(\lambda)$.

Also, as shown in Lemma 3, a left eigenvector corresponding to a nonzero eigenvalue can be determined by finding a vector in the left null space of the matrix $\sum_{j=0}^{\infty} \lambda^j A_j$. Thus the spectral representation of rate matrix \mathbf{R} can be found using the results in Section 3.

In [RKC], the problem of computation of rate matrix \mathbf{R} as a linear programming problem. Using a similar approach can be utilized for computation of \mathbf{R} , the minimal non-negative solution of a matrix power series equation.

When the rate matrix \mathbf{R} (a minimal non-negative solution of matrix polynomial equation) is not diagonalizable, method for computation of all generalized left eigenvectors and thus the Jordan canonical form representation of \mathbf{R} is discussed in [RKC]. Even in the matrix power series case, similar method can be derived. Details are avoided for brevity.

5 Computation of Rate Matrix of a G/M/1-Type Markov Chain:

Consider a G/M/1-type Markov chain X_n . The transition probability matrix, \mathbf{P} is of the form

$$P = \begin{bmatrix} D_0 & C_0 & 0 & 0 & 0 & 0 & \dots \\ D_1 & C_1 & C_0 & 0 & 0 & 0 & \dots \\ D_2 & C_2 & C_1 & C_0 & 0 & 0 & \dots \\ D_3 & C_3 & C_2 & C_1 & C_0 & 0 & \dots \\ D_4 & C_4 & C_3 & C_2 & C_1 & C_0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \dots\dots\dots(5.1)$$

The G/M/1-type Markov chain X_n is assumed to be irreducible and positive recurrent.

The equilibrium probabilities of the states on level n , \overline{v}_n are related to those at level $n+1$, through a matrix geometric recursion

$$\overline{v}_{n+1} = \overline{v}_n R'. \dots\dots\dots(5.2)$$

The rate matrix R' is the minimal non-negative solution of

$$\sum_{i=0}^{\infty} R'^i C_i = R' \dots\dots\dots(5.3)$$

Where C_i matrices are those in the transition probability matrix \mathbf{P} . Defining $\overline{C}_i = C_i$ for $0 \leq i \leq \infty, i \neq 1$ and $\overline{C}_1 = C_1 - I$ and substituting in (5.3), we have

$$\sum_{i=0}^{\infty} R'^i \overline{C}_i = 0. \dots\dots\dots(5.4)$$

It can be noted that each \overline{C}_i for $i \neq 1$ is nonnegative and that \overline{C}_1 has negative diagonal elements and non-negative off-diagonal elements. Thus \overline{C}_i 's have the same properties as A_i 's.

Since (5.4) is also of the same form as (3.1), the rate matrix R' of G/M/1-type Markov chain has many properties which are analogous to those of \mathbf{R} and thus can be computed using similar techniques as in the previous section. Details are avoided for the sake of brevity.

Now let us consider M/G/1-type Markov chains. In the analysis of such Markov chains, a matrix power series equation arises. As shown in [RKC], computation of minimal non-negative solution of such matrix power series equation can be carried out using the techniques discussed (in Sections 3 and 4) for G/M/1-type Markov chains. Details follow the same method as discussed in [RKC].

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