Acyclic Vertex Coloring of Graphs of Maximum Degree 6

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Kishore Kothapalli, V.Ch.Venkaiah, Satish Varagani

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Centre for Agriculture and Rural Development
International Institute of Information Technology
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Satish Varagani and V. Ch. Venkaiah


Kishore Yadav and Kishore Kothapalli

International Institute of Information Technology, Hyderabad, India – 500 032.

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1 Introduction

A proper coloring is said to be acyclic if the coloring does not induce any bichromatic cycles. The acyclic chromatic number of a graph \( G \) is denoted \( a(G) \), and is the minimum number of colors required to acyclically color the vertices of \( G \). The concept of acyclic coloring of a graph was introduced by Grünbaum [7] and is further studied in the last two decades in several works (cf.[3] and the references therein). Determining \( a(G) \) is a hard problem even theoretically as Kostochka [8] proved that it is an NP-complete problem to decide for a given arbitrary graph \( G \) whether \( a(G) \leq 3 \).

Using the probabilistic method, it was shown by Alon et al. [3] that any graph of maximum degree \( \Delta \) can be acyclically colored using \( O(\Delta^{4/3}) \) colors, thus showing that \( a(G) \leq O(\Delta^{4/3}) \). They further showed that a greedy algorithm exists to acyclically color any graph \( G \) with maximum degree \( \Delta \) using \( \Delta^2 + 1 \) colors. This was later improved by Albertson et al. [2] to show that \( a(G) \leq \Delta(\Delta - 1) + 2 \) for \( \Delta \geq 4 \).

Focusing on the family of graphs with a small maximum degree \( \Delta \), it was proved by Skulrattanakulchai [9] that \( a(G) \leq 4 \) for any graph of maximum degree 3. Burnstein [5] showed that \( a(G) \leq 5 \) for any graph of degree maximum

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2 Email: {vsatish@research.,kyadav@students.,kkishore@, venkaiah@iiit.ac.in
4. The work of Skulrattanakulchai was extended by Fertin and Raspaud [6] to show that it is possible to acyclically vertex color a graph $G$ of maximum degree $\Delta$ using at most $\Delta(\Delta - 1)/2$ colors. Recently, Yadav et al. [10] extended the work of Skulrattanakulchai [9] to show that any graph of maximum degree 5 can be colored using at most 8 colors.

In this paper, we show that any graph of maximum degree 6 can be acyclically colored using at most 12 colors. Our result thus improves the state-of-the-art for the considered family from 15 colors [6] to 12. We show the following theorem.

**Theorem 1.1 (Main Theorem)** The vertices of any graph $G$ of degree at most 6 can be acyclically colored using 12 colors in $O(n)$ time, where $n$ is the number of vertices.

### 2 Proof of the Main Theorem

Below, we first introduce the notation that is used in the rest of the paper. For a positive integer $k$, $\{k\}$ refers to the set of positive integers $\{1, 2, ..., k\}$. For $W \subseteq V(G)$, $N(W)$ is the set of all vertices in $V(G) \setminus W$ that are adjacent to some vertex in $W$. We borrow some notation from [9] and repeat it for the sake of completeness. A **partial coloring** is an assignment of colors to a subset of $V(G)$ such that the colored vertices induce a graph with an acyclic coloring. Suppose $G$ has a partial coloring. Let $\alpha, \beta$ be any two colors. An **alternating $\alpha, \beta$-path** is a path in $G$ with each vertex colored either $\alpha$ or $\beta$. An **alternating path** is an alternating $\alpha, \beta$ path for some colors $\alpha, \beta$. Let $v$ be an uncolored vertex. A color $\alpha \in \{12\}$ is **available** for $v$ if no neighbor of $v$ is colored $\alpha$. A color $\alpha \in \{12\}$ is **feasible** for $v$ if assigning color $\alpha$ to $v$ still results in a partial coloring. Let $C_v$ be a cycle in $G$ containing vertex $v$. A cycle $C_v$ is $\alpha, \beta$-dangerous if $C_v - v$ is $\alpha, \beta$-alternating path of even length. A cycle $C_v$ is dangerous if it is $\alpha, \beta$-dangerous for some colors $\alpha, \beta$. We call a vertex $v$ as a **single vertex** if all its colored neighbors receive distinct colors. The notion of a single vertex is useful because recoloring is easy at single vertices.

We prove the main theorem by extending a partial coloring to one vertex $v$ at a time. During this process, in some scenarios it is required that we recolor some of the vertices already colored so as to make a color feasible for the vertex which we try to color. However, note that this recoloring, if required, is limited to the neighborhood of the neighbors of $v$, in all cases. Specifically, we show the following lemmata which result in Theorem 1.1.

**Lemma 2.1** Let $\pi$ be any partial coloring of $G$ using colors in $\{12\}$ and let $v$ be any uncolored vertex. If $v$ has less than 4 colored neighbors, then there
exists a color \( \alpha \in \{12\} \) feasible for \( v \).

**Lemma 2.2** Let \( \pi \) be any partial coloring of \( G \) using colors in \( \{12\} \) and let \( v \) be any uncolored vertex. If \( v \) has exactly either four or five colored neighbors, then there exists a partial coloring \( \pi_1 \) of \( G \) using colors in \( \{12\} \) and a color \( \alpha \in \{12\} \) so that \( \pi_1 \) has the same domain as \( \pi \), \( \pi(x) \neq \pi_1(x) \) implies \( x \in N(v) \) and \( \alpha \) is feasible for \( v \) under \( \pi_1 \).

**Lemma 2.3** Let \( \pi \) be any partial coloring of \( G \) using colors in \( \{12\} \) and let \( v \) be any uncolored vertex. If \( v \) has six colored neighbors, then there exists a partial coloring \( \pi_1 \) of \( G \) using colors in \( \{12\} \) and a color \( \alpha \in \{12\} \) so that \( \pi_1 \) has the same domain as \( \pi \), \( \pi(x) \neq \pi_1(x) \) implies \( x \in N(v) \) or \( x \in N(N(v)) \), and \( \alpha \) is feasible for \( v \) under \( \pi_1 \).

Hence, Theorem 1.1 follows from the above lemmata. Below, we sketch the proof of Lemma 2.3. \(^3\)

2.1 Proof of Lemma 2.3

Before we prove Lemma 2.3, we start with a weaker statement with respect to the number of colors. We gradually reduce the number of colors that we require so as to arrive at Lemma 2.3.

Let us suppose that \( v \) is a uncolored vertex we are trying to color and that \( v \) has six colored neighbors. Let \( D(v) \) be the set of neighbors of \( v \) so that they are colored with a color that appears at least twice in the neighborhood of \( v \). A naive bound on the number of colors required can be obtained as follows. The number of colors that are not feasible for \( v \) is at most the number of colors used in the neighborhood of \( v \) plus the number of colored involved in dangerous cycles. Let us denote the former as \( C_N(v) \) and the latter as \( C_D(v) \). The number of dangerous cycles through \( v \) is at most \( \lfloor |D(v)|/2 \rfloor (\Delta - 1) \). Thus, \( C_D(v) \leq \frac{6}{2} \times 5 = 15 \). Using this bound on \( C_D(v) \) and noting that \( C_N(v) \leq 6 \), it can be seen that any graph of degree 6 can be acyclically vertex colored in no more than \( C_N(v) + C_D(v) + 1 = 22 \) colors. To improve this bound, we immediately note that \( C_N(v) \) and \( C_D(v) \) are related as follows. If \( |D(v)| \geq 5 \), then \( C_N(v) \leq 3 \). Similarly, if \( |D(v)| \geq 2 \), then \( C_N(v) \leq 4 \). Thus, we can improve the bound on the number of colors required to \( \max\{3 + (30/2), 6 + (10/2), 4 + (20/2)\} + 1 = 19 \) colors. The maximum in the above expression is related to the presence of single vertices as only single vertices contribute 5 to \( C_D(v) \). We use this observation to reduce the above bound on the number of colors required.

\(^3\) Proofs omitted due to space constraints can be found in [11].
Notice that if any vertex, say \( w \in D(v) \) is a single vertex, it can however be recolored, so that after recoloring \( w \not\in D(v) \), as follows. There are at most \( C_N(w) + C_D(w) = 5 + 0 \) colors that are not feasible for recoloring \( w \). So, there are at least 7 feasible colors for recoloring \( w \) when we try to color \( G \) with 12 colors. At least one of these does not appear in \( C_N(v) \). So, we can recolor \( w \) using that color. Now, as \( w \not\in D(v) \), the bound on the number of colors that are not feasible for \( v \) can be improved to \( \max\{3 + \frac{6 \times 3}{2}, 6 + (8/2), 4 + (16/2)\} = 15 \). Hence any graph of degree 6 can be acyclically vertex colored with at most 16 colors. We wish to add at this point that this is the result of Fertin and Raspaud [6] when \( \Delta = 6 \).

To further improve the bound, we make the following observation. Notice that the maximum in the above expressions is occurring when \( |D(v)| = 6 \). When \( |D(v)| = 6 \), we can observe that if all vertices in \( D(v) \) have at most three colors in their neighborhood, then \( C_N(v) \leq 3 \) and \( C_D(v) \leq 9 \). We can color \( G \) using 13 colors. If any vertex \( w \in D(v) \) has 4 colors in its neighborhood then we can reduce size of \( D(v) \) by using 12 colors as follows. There are, at most nine (apply \( C_N(w) + C_D(w) \)) colors are not feasible to recolor \( w \). Then, there are three colors that are feasible for \( w \). If \( |C_N(v)| = 2 \), then we can recolor \( w \) with the third color, thereby reducing the size of \( D(v) \). If \( |C_N(v)| = 3 \) then recolor \( w \) using a feasible color different from the color of \( w \). We conclude that if \( |D(v)| = 6 \), we can reduce \( |D(v)| \leq 5 \) or number of colors that are not feasible for \( v \) is at most \( 3 + (6 \times 3/2) = 12 \).

Similarly, if \( |D(v)| \leq 5 \), it can be shown that after a suitable recoloring of a vertex in \( D(v) \), 13 colors suffice to acyclically color any graph of degree 6. However, Lemma 2.3 states that 12 colors suffice. For this, we need to be use a more detailed analysis where one has to recolor vertices in \( N(N(v)) \). Below, we provide a sketch of the same.

**Proof Sketch.** Let \( v \) be the vertex that we extend the partial coloring to. We consider two cases in this proof sketch. The proof for other cases can be found in [11].

- **Assume without loss of generality that** \( N(v) = \{w, x, y, z, t, u\} \) and that \( \pi(w) = \pi(x) = 1, \pi(y) = \pi(z) = 2, \) and \( \pi(t) = \pi(u) = 3 \). Notice that \( v \) has 9 available colors and all of them can be involved in dangerous \( C_v \) cycles. So, no color may be feasible for \( v \). If any of the vertices in \( \{w, x, y, z, t, u\} \) are single, then we recolor a single vertex thereby reducing \( |D(v)| \) to 4. Otherwise, we make the following case distinction.
- **If any of** \( \{w, x, y, z, t, u\} \) **has four differently colored neighbors. Let it be**
If two neighbors of \( w \) have the same color and another two are colored with the same, but different from above, color, and the other two are differently colored: Assume without loss of generality that \( w \) has neighbors colored with colors in \( \{4,5,6,7\} \) and the other neighbor is colored with a color among the colors \( \{4,5,6,7\} \). Let it be 7 and the alike neighbors of \( w \) be \( w_1 \) and \( w_2 \). If any of \( \{8,9,10,11,12\} \) is found missing in the neighborhood of \( w_1 \) or \( w_2 \) then define \( \pi_1 \) by setting \( \pi_1(w) = k \) where \( k \) is the color missing and \( \pi_1(s) = \pi(s) \) for all other colored vertices \( s \). Then \( \pi_1 \) is also the a partial coloring. Moreover, under \( \pi_1 \), \(|D(v)| \) reduces to four. If none of \( \{8,9,10,11,12\} \) is found missing in the neighborhood of \( w_1 \) and \( w_2 \), then define \( \pi_1 \) by setting \( \pi_1(w_1) = 2 \), \( \pi_1(w) = 9 \), and \( \pi_1 \) agrees with \( \pi \) at all other colored neighbors. Notice that \( \pi_1 \) is also a partial coloring and under \( \pi_1 \), \(|D(v)| \) reduces to four.

- If all of \( \{w, x, y, z, t, u\} \) have only three differently colored neighbors. In this case, it can be shown that there exists a color to recolor \( w \) by observing the colors in \( N(W) \). So, let \( \pi_1 \) be the partial coloring where \( w \) is recolored. Under \( \pi_1 \), \( D(v) \) reduces to at most 5. A feasible color for \( v \) can be found subsequently by further analysis [11].

- If all of \( \{w, x, y, z, t, u\} \) have less than three differently colored neighbors:

  In this case, we notice that there exists a feasible color for \( v \).

- If two neighbors of \( v \) have the same color and another two are colored with the same, but different from above, color, and the other two are differently colored: Assume without loss of generality that \( \pi(w) = \pi(x) = 1 \), \( \pi(y) = \pi(z) = 2 \), \( \pi(u) = 3 \), and \( \pi(t) = 4 \). If any of \( w, x, y, z \) is a single vertex, we can recolor a single vertex thereby reducing \(|D(v)| \) to 2. Otherwise, for no available color to be feasible for \( v \) there must be four 1, \( \beta \)-dangerous \( C_v \) cycles and four 2, \( \beta \)-dangerous \( C_v \) cycles. Assume without loss of generality that there are 1, 5-, 1, 6-, 1, 7-, 1, 8-, and 2, 9-, 2, 10-, 2, 11-, and 2, 12-dangerous \( C_v \) cycles. This implies that neighbors of \( w \) and \( x \) are colored with colors in \( \{5,6,7,8\} \). Let the like neighbors of \( w \) be \( w_1 \), \( w_2 \), and \( w_1 \) and \( w_2 \) are colored with color 5. If none of \( w_1 \) or \( w_2 \) have a neighbor colored with color 1, apart from \( w \), then the color 5 is feasible for \( v \) since there is no possibility of 1,5–dangerous \( C_v \) cycles. Hence, let \( w_1 \) have a neighbor colored with color 1. We distinguish between two cases. If any of the colors in \( \{9,10,11,12\} \) is missing in \( N(w_1) \), then we define \( \pi_1 \) by setting \( \pi_1(w) = k \), where \( k \) is missing in \( N(w_1) \), and \( \pi_1(s) = \pi(s) \) for all other colored vertices \( s \). Then, \( \pi_1 \) is also a partial coloring and under \( \pi_1 \), \(|D(v)| \) reduces to two. Otherwise, define \( \pi_1 \) as \( \pi_1(w) = 9 \) and \( \pi_1(w_1) = 3 \) and \( \pi_1 \) agrees with \( \pi \) at all other colored vertices. Under this definition of \( \pi_1 \), which is a partial coloring, \(|D(v)| \) reduces to two.

\[\Box\]
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References


