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A thesis on “Analysis of underlying topology on dynamical networks”

by

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I certify that the work contained in this thesis, titled “Analysis of underlying topology on dynamical networks” by Raina Arora has been carried out under my supervision and in my opinion, is fully adequate in scope and quality as a dissertation for the degree of Master of Science by Research in Computational Natural Sciences.

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Date
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Abstract

Two classes of networks that have been extensively studied in the dynamical analysis of spatially extended systems are: (i) regular networks, wherein each node interacts with a specified number of neighbouring nodes on geometrical lattices, and (ii) random networks, wherein each pair of nodes has a fixed probability of interacting with each other. Recently much attention is being focused on a class of network models that are neither strictly regular nor completely random, but are somewhere in between and exhibit properties of both. Such networks are called small-world networks. Examples for small-world networks are found to occur widely across biological (e.g., neural connection patterns), social (e.g., friendship network, co-authorship) and technological (e.g., the World Wide Web) domains. It has also been observed that a large number of real complex systems such as protein contact networks, protein-protein interaction networks, metabolic networks, exhibit power-law behaviour in their degree distribution, i.e., a few nodes have a very high degree. Such networks are referred to as scale-free networks. To model real physical and biological systems governed by dynamical processes, on such networks the nodes are defined by dynamical systems and the connection between them may be fixed or changing in response to the local dynamics of the nodes, e.g., neuronal excitations. Such networks are called ‘Dynamical Networks’. Thus, from the point of view of dynamical systems, we have a coupled system of dynamical equations, and we are interested in controlling or manipulating the resulting global dynamics of the system emerging from the interactions between the local dynamics of the individual elements. We propose to use concepts from Graph theory to analyze the coupling structure and its influence in attaining global control of the resulting dynamics in the event of the system exhibiting complex chaotic or excitable dynamics.

In an earlier study by Parekh et al, robust control (both global and local) of spatiotemporal dynamics on coupled map lattice (CML) models by applying a constant external perturbation or “pinning” has been shown. The advantage of this control approach is that neither any a priori knowledge of the system dynamics, such as stable or unstable fixed points and periodic orbits nor any modification/tracking of the system parameters/variables explicitly is required to suppress chaos. In this study we use the same control approach and analyse the role of connection topologies in achieving global control of the dynamics on small-world and scale-free networks. The analyses have been carried out for dynamics on the nodes governed by (a)
logistic map (a one-dimensional nonlinear map exhibiting a wide variety of dynamical behaviour including chaos), and (b) neuron map (a two-dimensional nonlinear map modelling an excitable system, e.g., neuron). The two dynamical systems have been modelled on four different network topologies, viz., regular, random, small-world and scale-free. Since it is practically infeasible to pin all the nodes of the network, here we exploit the geometrical/topological properties of networks to choose nodes for exerting control. We observe that in the case of small-world networks, wherein the connection topology is predominantly local, with a few long range interactions, regularly-spaced controllers provide better control for low pinning densities ~ 20% compared to hubs and other centrality measures which require pinning ~ 50% nodes at similar pinning strengths. The long-range connections only seem to have the effect of spatial noise on small-world network. However, in case of controlling dynamics on the scale-free network, centrality nodes do provide significant advantage in achieving global control. We observe that high-betweenness and high-closeness nodes at very low pinning densities ~ 10 – 15% suffice in suppressing spatiotemporal chaos.
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1 Introduction

1.1 Dynamical Systems and Chaos Theory

Dynamical systems are those systems that change in time. Mathematically, they can be presented as a concept to describe the time dependence of a point’s position in its ambient space. These systems are highly sensitive to the initial conditions. Even though they are deterministic (that is, fully determined by the initial conditions), long-term prediction of the dynamics of these systems becomes impossible because slight difference in the initial conditions can significantly alter the resulting dynamics (Kellert, 1993). This behaviour is called chaos. Examples of systems in which chaotic behaviour is observed include changes in weather phenomena, movement of satellites in the solar system, electrical circuits, lasers, oscillating chemical reactions, population growth in ecology, and dynamics of action potential in neurons (Sneyers, 1997). The theory of chaos is being also applied to study pathological conditions such as epileptic seizures and heart strokes, and to be able to predict seemingly random attacks by observing the initial conditions (Freeman, 1991).

In order to determine the predictability of dynamical systems, a mathematical concept called Lyapunov exponent is used. It is the rate of divergence from perturbed initial conditions. A positive exponent indicates that the system is chaotic while a negative value indicates stable or non-chaotic dynamics.

1.2 Synchronization and Control of Dynamics: Its Significance

Synchronization is encountered in a variety of natural phenomenon such as synchronous firing of neurons, adjustment of heart rate with respiration and/or locomotory rhythms, and different forms of cooperative behavior of insects, animals and even humans. Our surroundings are full of oscillating objects. Radio communication and electronic equipment, fireflies emitting sequences of light pulses, birds flapping their wings, a neural center that controls the contraction of the human heart, the heart itself, are all systems that share a common feature – they produce rhythms. These objects do not operate in isolation but interact with other objects in their environment. For example, a firefly is influenced by the light emission of the whole population; different centers of rhythmic brain activity may influence each other. This interaction causes an object to adjust its rhythm in conformity with the rhythms of other objects. As a result, insects in
a population emit acoustic or light pulses with a common rate, birds in a flock flap their wings
simultaneously, and the heart of a rapidly galloping horse contracts once per locomotory cycle.
This adjustment of rhythms, due to an interaction, is the essence of synchronization. Hence, we
see that a number of systemic and environmental factors govern the normal dynamics and the
synchronization of a system. It has also been found that perturbation in any of these factors is
responsible for altering the normal dynamics of the system. Defects in biochemical functioning
of a healthy neuronal tissue or an ectopic node of the heart reflect in their altered dynamics and
disturbance in their synchronization.

As an example, we study the case of a neuron network. We know that chaos is
omnipresent in the brain – in its somatosensory and olfactory cortices (Freeman, 1987). It has
been suggested that the quiescent state of the brain is that of chaos while during perception, i.e.,
when attention is focused on any sensory stimuli, brain activity becomes more periodic
(Freeman, 1991). In clinical conditions such as hallucinations prevalent during epileptic seizures,
it has been found that there is undesired control in the chaotic activity of the brain. The
stabilizing of an undesired periodic attractor results in erroneous recognition of sensory stimuli
even when such a stimulus is absent from the immediate environment of the epileptic subject
(Skarda and Freeman, 1987). Hence, it is of practical significance to control the dynamics of
such systems and to bring it back to its original state of chaos. There are other systems where
control of chaos, i.e., taking the system from chaotic to stable state, is important. These situations
are instabilities in laser, desynchronization in coupled chemical reactors, and epidemics in meta-
population. Besides these practical applications, the ability to control spatiotemporal dynamics
provides a unique capability to study otherwise inaccessible states of extended non-equilibrium
systems.

1.3 Real-World Networks: Modeled as Graphs

Recently, much work has been done on modeling real-world systems as graphs. Such
systems comprise a large number of nodes (or elements) linked with each other according to
specific connection topologies, and are seen to occur widely across the biological, social and
technological worlds (Barabasi, 2002; Newman, 2003). The links (or edges) between these nodes
represent the relationship connecting them. Examples range from the intra-cellular signaling
systems that consist of different kinds of molecules affecting each other via enzymatic reactions,
to the internet composed of servers around the world, which exchange enormous quantities of
information packets regularly (Dorogovtsev, 2003), and food webs which link large number of inter-dependent species via trophic relations. Great amount of useful information has also been derived on systems such as electron delocalized molecules, example benzene, by considering electrons and atomic orbitals as vertices and overlap between them as edges; and chemical molecules by considering atoms as vertices and bonds between them as edges (Gutman, 1986; Bonchev, 1991).

The existence of complex networks in various domains has been known for some time now, but the recent improvements in the computing power have provided impetus to the study of networks having large sizes and great complexity in connections. Also, it has been observed that systems that were physically considered very different from each other, actually share certain universal features in their connectivity patterns. Prior to these developments, the networks studied by scientists were either regular or random in their connectivity patterns. Regular networks are defined on a geometrical lattice where a node interacts with other nodes in its vicinity (or neighbourhood) while random networks have no such order in their connectivity. In fact, a node in a random network is connected to another with a fixed probability. It was found that many biological, social and technological networks lie somewhere between these two extremes and exhibit properties of both regular and random networks (Watts and Strogatz, 1998).

The term ‘small-world’ was coined to describe such networks. The neural network of C. elegans, coactive functional brain areas (Achard, 2006), the power grid of the western United States, the Indian railways system (Sen et al., 2003), and the collaboration graph of film actors are shown to be small-world networks. The peculiar feature of such networks is that a node can be reached from another node in the network by passing through a very small number of nodes. This property is reflected in these networks having small average path length (the arithmetic mean of the shortest distances between each pair of nodes). Owing to small average path length, infectious diseases spread more easily in small-world networks compared to regular networks, which have high average path length. Also, these networks are found to exhibit high clustering of nodes such that nodes in the vicinity of a node are very strongly connected amongst each other. This explains the clustering of specific types of amino acids important for structure, folding, and function. To interpolate between regular and random networks, the scientists, Watts and Strogatz, described a model that considers random rewiring procedure for the construction of a small-world network (Watts and Strogatz, 1998).
Soon after the concept of small-world networks came into being, it was also observed that a wide variety of systems such as the World Wide Web have the frequency distribution of number of connections of their nodes (called the degree distribution) exhibiting a power law (Barabasi, 1999). This further proved the fact that most real-world networks have topology that is neither regular (in which case all the nodes in the network should have had the same degree and the degree distribution should have been a delta function) nor random (in which case the degree distribution is Poisson). In both cases, the probability of a node having high degree will be significantly smaller than that indicated by the power law tail. In addition, it was also observed that there is a tendency of some nodes to be preferentially connected to some other nodes in the network. It was seen that in social networks, nodes having high degree tend to be preferentially connected with the other high degree nodes (called assortative mixing) while in biological and technological networks, nodes with high degree tend to connect to nodes with low degree (called disassortative mixing) (Newman, 2002).

1.4 Modeling Dynamical Networks

The theory of dynamical networks can be viewed as a combination of graph theory and non-linear dynamics. In Section 1.3, we described how various kinds of real-world systems from chemical, biological, and social domains can be modelled as graphs. Non-linear dynamics helps us in analysing the dynamics of participating entities or nodes of these systems and their interactions with their neighbouring elements. Depending upon the network being analysed, its study may involve considering the topology of the network changing in time such as in World Wide Web, or the dynamics on individual nodes evolving in time with the topology remaining fixed as in neuronal networks or both the topology and individual nodes evolving in time. Such systems are called Dynamical networks. In modelling dynamics of different types of systems, a variety of mathematical equations such as logistic map, Lorenz attractor, Henon map, and Chialvo’s neuron map have been proposed. Dynamics on each node of the network (represented as a coupled map lattice) is modelled by these maps (Kaneko, 1992). With the recent advancements in establishing the non-regular topology of the real-world networks, work has been done on studying the dependence of dynamics of such a network on its structure or the underlying topology. It has been found that structure affects the dynamics of a network and hence its function. Conversely, the functional criteria, \textit{i.e.}, the need for dynamic stability can
constrain certain features of the topology of the network. Hence, it is of practical significance to study the role that the underlying topology of the network plays in the dynamics of the system.

1.5 Synchronization and Control in Networks having Non-Regular Topology

A large number of studies have been carried out to understand the synchronization and control in a variety of systems including coupled map lattices by Kaneko (1998). Until recently, these systems were investigated under the assumption of regularity in their connectivity patterns, where the participating units were coupled only to the nearest neighbours. Lately, the topologies of real-world systems have been established as being non-regular (random, small-world, scale-free, hierarchical). A study by Jost et al (2004) has shown that systems with non-regular topologies represented as coupled map lattices, where each dynamics on each node is governed by a logistic map, synchronize over a wider range of parameters compared to networks with regular topology. A wide range of dynamical behaviour can be accessed by varying a parameter called coupling strength. (Jost, 2004) This parameter determines how tightly or loosely the participating nodes are coupled with each other, which in turn determine the flow of information from one node to another.

A similar study modelling neuronal networks as coupled map lattices; where dynamics on each node was governed by a neuron map (Chialvo, 1995) was carried out to study the effect of coupling strength in spatiotemporal dynamics (Jampa et al, 2007). They studied the role of topology in spatiotemporal dynamics by varying the probability with which the edges in a regular network were rewired. They observed that both regular and random networks synchronize for high values of coupling strength. However, at low values of coupling strength, regular networks become asynchronous while random networks still synchronize. This indicates that networks with non-regular topologies tend to synchronize over a wider range of parameter space compared to regular networks.

As discussed in Section 1.2, the ability to control/manipulate spatiotemporal dynamics has implications in real and artificial systems. Several approaches for achieving such control of dynamics have been proposed in the past. However, they require \textit{a priori} knowledge about the system dynamics. For this, sensitive and sophisticated experimental methodology is required making it difficult to implement these techniques. One such approach to manipulate spatiotemporal dynamics is to apply a constant perturbation or ‘pinning’ signal (Parekh et al.,
1998). The advantage with this approach is that it is not system-specific. The sign and the strength of the signal alone can determine the control of spatiotemporal dynamics to desired target states without requiring any prior information of the system parameters and their modification. In studying the control of spatiotemporal dynamics in different types of topological networks, in this study, we propose to exploit the unique features of the topologies of these networks. Hence, we wish to investigate the role that the underlying topology of the network plays in achieving control of dynamics.

1.6 Organization of the thesis

The thesis has been divided into three chapters. The first part of Chapter 2 gives an introduction to graph theory. The models used for the construction of four different kinds of topological networks used in our study (regular, small-world, scale-free and random) and their global properties have been described. The second part of this chapter introduces the concept of modeling dynamics on these networks. The dynamics of two maps, viz., logistic map and neuron map followed by control of dynamics by applying external perturbation or pinning has been described. Chapter 3 contains the results for synchronization and control of the four topological networks where the dynamics on each node has been modeled by (1) logistic map and 2) neuron map. Results for control by pinning all nodes, regularly spaced nodes, randomly chosen nodes and certain critical nodes have been presented for both the cases. Finally, in Chapter 4, we present our conclusions of the analysis.
2 Methods And Materials

This chapter is organized as follows. We first present an introduction to graph theory and briefly describe the construction of four different topological graphs studied in this thesis, namely, regular, small-world, scale-free and random networks. This is followed by the matrix representation of the graph and various topological properties and centrality measures used for the characterization of these graphs. Next we introduce the concept of dynamical networks and give a brief description of the construction of these networks. The two dynamical networks analyzed in this thesis are constructed with dynamical nodes modeled by difference maps: (1) logistic map (1-d map known to exhibit a wide variety of spatiotemporal behaviour) and, (2) neuron map (2-d map to model the neuronal excitations in the brain). Finally, the calculation of Lyapunov exponents to characterize the chaotic dynamics and the control approach applied to suppress/enhance spatiotemporal chaos in the dynamical networks is discussed.

2.1 Introduction to Graph Theory

Graph theory is a field of mathematics that is used to model relations between pairs of objects. Different types of objects and relationships between them are observed in a number of systems known to exist in domains such as chemistry, physics, sociology, biology, etc. These systems can be modeled as graphs. A graph can be defined as a collection of objects joined by links based on the relationship between these objects. The interconnected objects are represented by vertices or nodes, and the links that connect pairs of vertices are called edges. Examples of systems that can be represented as graphs include molecules and chemical reactions, electrical circuits giving relationships between different components, internet connecting servers around the world where servers are the nodes and connections between them are the edges, the power grid, the airport network describing the connecting flights between airports, the Indian railways system, the collaboration graph of film actors where the relationship between actors is defined by the movies shared by the actors, citation network, metabolic networks, gene regulatory networks whose vertices are proteins and genes and chemical interaction between them are the relations, and food webs (Barabasi and Albert, 2002; Newman, 2003). Depending upon the nature of relationships between the participating nodes in the network, the edges between them can be
directed or undirected. In the case of a chemical compound, atoms are the nodes and the bonds between them are the undirected edges since the bonds are non-directional. In the case of a network of food web, nodes represent organisms and the edges between them depend upon the prey-predator relationship, which is directed from predator to prey. The graph for the citation network will have each node representing a paper in literature and edges between pair of nodes based on whether a paper cites another paper as a reference. Thus any group of entities which can be linked to each other by some association can be represented by a graph.

Mathematically, a Graph is defined as $G = (V, E)$ where $V$ is a set of vertices (or nodes), and $E$ is a set of edges. An edge is drawn between a pair of nodes if a relation can be defined between them and then the two vertices are said to be adjacent. For example, for the graph shown in the Figure 2.1, the vertex set is $V = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ and the edge set is $E = \{1-4, 2-4, 3-4, 4-5, 5-6, 6-7, 6-8, 6-9, 6-10\}$.

![Figure 2.1: An example of a Graph, $G$ showing numerically labeled nodes and edges between them.](image)

The nodes and the connectivity information between them can be represented mathematically in the form of a matrix. Different types of matrices have been used to capture the topological properties of the graph. The most extensively studied matrices for representing a graph are the degree, adjacency and Laplacian matrices and are described below.

**Degree Matrix:** It is one of the simplest matrix representations of a graph giving the connectivity information of all the nodes in the graph. It is a diagonal matrix with the diagonal elements representing the degree of each node. Thus, the elements of the degree matrix, $\text{deg}_{ij}$, are given by:
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\[
\text{deg}_{ij} = \begin{cases} 
  k_i & \text{if } i = j \\
  0 & \text{if } i \neq j 
\end{cases}
\]

where \( k_i \) is the degree of the node \( i \), i.e. the number of nodes it is directly connected to. The larger the degree, the more important the node is because of its better connectivity in the network. For example, in Fig. 2.1, nodes 4 and 6 have degree 4 and 5 respectively, and these are the high degree nodes, or “hubs” of the graph \( G \). The degree matrix, \( D(G) \) of the graph \( G \) in Fig. 2.1 is:

\[
D(G) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

**Adjacency Matrix:** The most widely used matrix for representing a graph is the adjacency matrix. It captures the connectivity information of the entire graph. An adjacency matrix, \( A(G) \), of a graph having \( N \) nodes is a symmetric square matrix of dimensions \( [N \times N] \) defined as follows. Each element, \( a_{ij} \), of the adjacency matrix is ‘1’ if nodes, \( i \) and \( j \) are connected to each other and ‘0’ if they are not connected. Since a node cannot have an edge to itself, all the diagonal elements are 0. Mathematically, it is defined as:

\[
a_{ij} = \begin{cases} 
  1 & \text{if } e_{ij} \in E \\
  0 & \text{if } i = j \text{ or } e_{ij} \notin E
\end{cases}
\]

where \( e_{ij} \) is the edge between nodes \( i \) and \( j \) and \( E \) is the set containing all the edges in the graph. The adjacency matrix, \( A(G) \) for the graph in Fig. 2.1 is:
Laplacian Matrix: It explicitly incorporates the information contained in both the adjacency and the degree matrices and is defined as:

\[
L(G) = D(G) - A(G)
\]

where \(D(G)\) is the degree matrix and \(A(G)\) the adjacency matrix of the graph, \(G\). Each element \(l_{ij}\) of the Laplacian matrix is thus defined as:

\[
l_{ij} = \begin{cases} 
 k_i & \text{if } i = j \\
 -1 & \text{if } i \neq j, e_{ij} \in E \\
 0 & \text{otherwise}
\end{cases}
\]

Thus, for the graph \(G\) in Fig. 2.1:

\[
L(G) = \begin{bmatrix}
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 4 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 5 & -1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
2.2 Construction of Topological Networks

To model real systems, various connection topologies have been defined and extensively studied. Below we describe the construction and properties of the four graph topologies considered in this study, viz., regular, small-world, random and scale-free.

Regular Network: The simplest example of a network or a graph is one in which each node is connected to a fixed number of neighbours. A representative example of such a network is shown in Fig. 2.2, on a circular lattice, wherein each node $i$ is connected to three nodes on either side i.e. to nodes $i+1$, $i+2$, $i+3$, $i-1$, $i-2$, $i-3$. Such a regular pattern of connectivity is observed in a very few real systems such as arrangement of atoms in a crystal, viz., sodium chloride, array of Josephson junctions, etc.

![Regular Network](image)

**Fig. 2.2:** A representative regular network of size $N = 20$ and $k = 6$ drawn in Pajek (Batageli and Mrvar, 2003)

Random Network: To explain complex connection topology observed in most real systems, Erdos and Renyi proposed that a graph be constructed by connecting nodes randomly or a graph is chosen uniformly at random from a collection of all graphs having a particular number of nodes and edges. Here we have used the approach used by Watts and Strogatz for the construction of random networks (Watts and Strogatz, 1998). Starting with a regular network, each edge is rewired with a certain probability, $p_r$. For $p_r = 1$, all the edges in the network are rewired and the network generated has random connection topology. This is shown in Fig. 2.3.
Small-world network: The two networks discussed above represent two extreme situations of connection topology. However, most real-world systems neither exhibit strictly regular patterns in their connectivity nor strictly random, but somewhere in between. Watts and Strogatz (1998) proposed a model to describe the connection topology in which few nodes exhibit long-range (random) connections in a network of mainly short-range interactions. Starting with a regular network on a one-dimensional lattice of size $N$ with each node connected to $k$ nearest neighbours (i.e., $k/2$ on either side), every edge is rewired with a certain probability, $p_r$, to a randomly chosen node. For very small probability values, $p_r \ll 1$, the network exhibits few long-range interactions. Watts and Strogatz coined the term “small-world” to describe such networks. In Fig. 2.4, a small-world network of size $N = 20$ constructed with $p_r = 0.09$ is shown.

A number of variants to this approach for the construction of random networks have also been proposed. In the approach discussed above, an edge $e_{ij}$ is rewired from $j$ to another destination node, $k$ chosen randomly such that the new edge is $e_{ik}$. Instead of this approach, one can have two nodes, say, $l$ and $m$ being chosen randomly to replace an existing edge $e_{ij}$ by an altogether new edge, $e_{lm}$. Another approach proposed is to keep the nearest neighbour interactions in the regular lattice intact and introduce a few new connections between randomly chosen nodes (Newman and Strogatz, 2001).

The notion of small-world in such networks comes from the observation that one can reach from one node to another in a small number of steps. Some degree of randomness in the spatial coupling compared to strict nearest-neighbour interactions is common in many systems, viz., World Wide Web, friendship networks, road maps, gene regulatory networks, food chains, neural networks and metabolic networks to name a few.
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Fig. 2.4: A representative small-world network of size $N = 20$ and $p_r = 0.09$ constructed by Watts-Strogatz approach (drawn in Pajek).

Scale-Free Network: It has been observed that many networks apart from exhibiting long-range connections also exhibit a unique feature in their connectivity i.e., a few nodes have very high degree, e.g., World Wide Web, neural networks, citation networks and disease networks. For example, in the case of World Wide Web, every time a new web page is added, the chances that it gets linked to an earlier frequently accessed page are very high. As a result most popular web pages get a large number of links to the newly added pages. Similarly, a review article would be cited most often in citation networks. Barabasi and Albert (1999) proposed an algorithm to model such systems. The characteristic features of this algorithm are growth and preferential attachment. Starting initially with $m_0$ nodes randomly connected to each other, the network is built by adding a new node at every time step. The new node is connected to $m$ already existing nodes by preferential attachment, i.e., with a probability proportional to its degree. This means a new node gets connected to an existing node, $i$, with the probability given by

$$
\prod (k_i) = \frac{k_i}{\sum_{j\neq i,j=1}^N k_j}
$$

2.1

In Fig. 2.5 is shown a representative scale-free network constructed by starting initially with $m_0 = 3$ nodes and adding $m = 2$ new connections at each step till the network size is $N = 50$ and the total number of edges is 102. The network is clearly inhomogeneous: while the majority of nodes have two or three links, a few nodes have a large number of links. For clarity, the high degree nodes have been pulled to the periphery.
For our analysis, we have constructed the four networks of size $N = 1000$ nodes as discussed above. In the regular network, the number of nearest neighbour interactions considered on either side is three, *i.e.*, $k = 6$. Small-world network is constructed starting with this regular network and rewiring every edge with the rewiring probability, $p_r = 0.09$ by Watts and Strogatz approach (1998). In the network thus constructed, the average degree of the network is 6, with the highest and lowest degree being 9 and 4 respectively. Random network has been constructed by rewiring all the edges in the regular network to a randomly chosen node, *i.e.*, $p_r = 1$. The average degree in this case is also 6. Scale-free network has been constructed by starting with a network of size, $m_0 = 5$ and adding one node and three new connections ($m = 3$) at each time step by the algorithm proposed by Barabasi and Albert (1999) till the network size is $N = 1000$. The scale-free network thus constructed has few nodes with very high degree ($\sim 60$) and a large number of nodes having small degree ($\sim 3$), with the average degree being 6. Care has been taken to keep the average connectivity the same in all the four cases for comparison purposes.

To analyse the effect of average connectivity, we have also constructed networks with average degree four, *i.e.* starting with 4 nearest neighbours on a regular network. In the case of scale-free network, average degree 4 was obtained by adding 2 new connections at every time step, *i.e.*, $m = 2$. To analyse the effect of the number of long-range interactions, small-world networks have also been constructed with two different rewiring probabilities, $p_r = 0.01$ and $p_r = 0.05$. 

![Figure 2.5: A representative scale-free network for $N = 50$ nodes constructed by Barabasi-Albert approach (drawn in Pajek)](image-url)
2.3 Graph Properties

The three most robust measures for characterizing graphs are degree, average clustering coefficient and path length. These measures characterize a node in terms of its connectivity, the connectivity of its neighbourhood and the overall connectivity of the graph, respectively. Another measure called the diameter gives an idea about the compactness of the network. A brief description of these properties is given below.

**Degree:** It is a measure of the connectivity of a node and is defined as the number of nodes to which it is directly connected (i.e., the number of neighbours of node, \(i\)):

\[
k_i = \sum_{j, j \sim i} a_{ij}
\]

where \(a_{ij}\) are the elements of the adjacency matrix, and \(j \sim i\) indicates that node \(j\) is a neighbour of node \(i\). The higher the degree, the more important the node is to the existence of the network. For example, in the case of World Wide Web, where an edge exists between two websites if one contains a link to the other, the high degree nodes will be websites which are very popular and frequently accessed (such as Wikipedia). Thus, based on their connectivity in the network, nodes critical to the stability/existence of the network can be identified.

**Characteristic Path Length:** Among all possible paths that exist between a pair of nodes, the geodesic or the shortest path is defined as the route having the minimum number of nodes between them. If \(d(i,j)\) is the length of the shortest path between a pair of nodes, \(i\) and \(j\), then the characteristic path length, \(L\), is defined as \(d(i,j)\) averaged over all \(\binom{N}{2}\) pairs of nodes (Watts and Strogatz, 1998):

\[
L = \frac{\sum_{i,j} d(i,j)}{1/2[N(N-1)]}
\]

where \(N\) is the number of nodes in the graph. For example, in the case of World Wide Web, this measure gives an idea of how many clicks one requires to reach one website from another. It is a measure of the global property of the network and indicates how well-connected a graph is. It also reflects the overall efficiency of the network, i.e., the ease with which information can be transferred between individual entities in the network. Consider nodes 1 and 4 in Fig. 2.6. To reach from node 1 to node 4, various possible paths are \(1 \rightarrow 3 \rightarrow 5 \rightarrow 2 \rightarrow 3 \rightarrow 4\), \(1 \rightarrow 2 \rightarrow 3 \rightarrow 4\),
and $1 \rightarrow 3 \rightarrow 4$. Assuming equal weight of 1 between each pair of adjacent nodes, the lengths of these paths are 5, 3, and 2 respectively. Thus the shortest path length between nodes 1 and 4 is 2. In this way, the shortest path length between each pair of nodes is calculated and averaged over all pairs of nodes to obtain the characteristic path length, i.e., $L = 2.9$ for the graph given in Fig. 2.6.

![An example graph](image)

**Fig. 2.6:** An example graph

**Average Clustering coefficient:** It gives information about how strongly connected the neighbourhood of a particular node is in the network (Costa *et al.*, 2007). The local clustering coefficient $C_i$ for a node, $i$, is given by the proportion of links between the vertices within its neighbourhood divided by the number of links that could possibly exist between them. The neighbourhood, $N_i$, of a node, $i$ having edges $e_{ij}$ with neighbouring nodes, $j$, is defined as:

$$N_i = \{ j : e_{ij} \in E \}$$  \hspace{1cm} 2.4

where $E$ is the set containing all the edges in the graph. The local clustering coefficient, $C_i$, of a node, $i$ is thus given by:

$$C_i = \frac{|\{e_{jk}\}|}{1/2[k_i(k_i - 1)]} : j, k \in N_i, e_{jk} \in E$$  \hspace{1cm} 2.5

For the graph given in Fig. 2.6, the neighbourhood of node 3 is $N_3 = \{1, 2, 4, 5\}$. Out of these four nodes, nodes 1 & 2 and nodes 2 & 5 have an edge between them. So, $|\{e_{jk}\}| = 2$, $k_i = 4$, thus, $C_3 = 2/((4\times3)/2) = 1/3$. The average clustering coefficient of the graph is calculated by averaging over the clustering coefficients of all the nodes in the graph:
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\[ C = \frac{1}{N} \sum_i C_i \]

and its value is 0.60 for the graph in Fig. 2.6.

**Diameter:** It is defined as the maximum among all the shortest path lengths between \((i, j)\) pairs in a network:

\[ D = \max_{ij} d(i, j) \tag{2.6} \]

A large value of the diameter indicates that the graph is less compact and one requires a large number of steps to reach the two farthest nodes in the graph. As is clear from Table 2.1, the regular network, having an elongated structure has a very high value of diameter while scale-free and random networks are most compact, requiring very few hops between two farthest nodes.

**Global Efficiency:** The concept of network efficiency has been used to characterize the properties of small-world and scale-free networks to study the effects of errors and attacks on the global and local properties of networks. It has been shown by Latora and Marchiori (2001) that the global efficiency of a graph, \(G\) can be used to describe the response of complex networks to external factors and is given by

\[ E_{\text{global}} = \frac{1}{N(N-1)} \sum_{i,j} 1/d(i, j) \tag{2.7} \]

where \(d(i,j)\) is the shortest path between a pair of nodes \(i\) and \(j\). It may be noted that even when two nodes have no path between them, \(i.e., d_{ij} = +\infty\), \(E_{\text{global}}\) is determinate. We show here that the effect on transmission of information through the small-world and scale-free networks on removal of various centrality nodes can help in assessing the efficacy of achieving control by pinning.

The transmission efficiency on the four topological networks is summarized in Table 2.1. As is clear from the table, regular network with a large value of the characteristic path length is the least efficient. On introduction of a few long-range connections in small-world network, the global efficiency of the network is increased by a factor of 5 compared to the regular network. This is a useful measure to compute transmission in a network which is not fully connected. It can be used to analyse the importance of a node in maintaining the flow of transmission in the network. As expected, scale-free networks are more efficient than random networks since they
have some nodes with very high degree which provide shorter transmission paths to other nodes through them.

### 2.3.1 Analysis of the topological properties of four networks

To identify the range of rewiring probability, $p_r$ in the Watts-Strogatz model for which the network exhibits small-world behavior, we analyzed the average clustering coefficient and the characteristic path length as a function of $p_r$. The results are depicted in Fig. 2.7 for ten realizations of the network topologies. It is clear from the figure that for $p_r$ values between 0.005 and 0.1 (region enclosed between two parallel lines), the average clustering coefficient is as high as that for a regular network while the characteristic path length is close to that of a random network. That is, the network exhibits small-world behavior for $p_r$ values within this range and is characterized by the following conditions:

$$C_{reg} \sim C_{sw} >> C_{rand}$$
$$L_{reg} >> L_{sw} \sim L_{rand}$$

![Fig. 2.7: The plot of average clustering coefficient ($C(p_r)$) and characteristic path length ($L(p_r)$) is shown as a function of rewiring probability, $p_r$, plotted for ten realizations. $C(p_r)$ and $L(p_r)$ have been normalized by their values at $p_r = 0$.](image)

We summarize in Table 2.1 the values of characteristic path length, average clustering coefficient, diameter and global efficiency for the four topological networks.
Table 2.1: The characteristic path length ($L$), average clustering coefficient ($C$), diameter ($D$) and global efficiency ($E_{\text{global}}$) for the four topological networks shown.

<table>
<thead>
<tr>
<th>Network</th>
<th>$L$</th>
<th>$C$</th>
<th>$D$</th>
<th>$E_{\text{global}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regular</td>
<td>83.75</td>
<td>0.6</td>
<td>167</td>
<td>0.0342</td>
</tr>
<tr>
<td>Small-World</td>
<td>6.41</td>
<td>0.56</td>
<td>12</td>
<td>0.1726</td>
</tr>
<tr>
<td>Scale-Free</td>
<td>3.54</td>
<td>0.058</td>
<td>6</td>
<td>0.2978</td>
</tr>
<tr>
<td>Random</td>
<td>4.10</td>
<td>0.004</td>
<td>6</td>
<td>0.2589</td>
</tr>
</tbody>
</table>

It is clear that the characteristic path length, $L$, is very high for a regular network compared to the other three networks. This is because of only nearest neighbour short-range connections between the nodes in the case of regular network. As a consequence, a large number of steps are required to reach from one end of the lattice to the other. This results in a large value of the diameter, $D = 167$, and very small value of global transmission efficiency for this network. It has a high clustering coefficient since the nodes lying in the immediate neighbourhood of a particular node are also connected with each other. As a consequence of random rewiring of a few edges in the small-world network, long-range interactions are introduced which result in a drastic reduction of the average path length to $\sim 6$. In this case a maximum of 12 hops between two farthest nodes in the network is observed. The small values of the average path length and the diameter are the defining features of a “small-world” nature of this network. The reduction in the path length is seen to increase the global efficiency of the network 5-fold. Since majority of the nodes in the small-world network have short-range nearest-neighbour connections, the clustering coefficient is high and comparable to that of a regular network. As more and more edges are rewired, both the clustering coefficient and the path length decrease, while the global efficiency increases and for rewiring probability, $p_r = 1$, corresponding to random connection topology, the clustering coefficient and the path length are the minimum.

For the scale-free network, the characteristic path length is the lowest among the four topological networks. This is because few nodes having very high degree provide shorter access routes to the other nodes in the network; this is also reflected in a low value of the diameter, 6. As a consequence, this network also has the highest value of global efficiency. The clustering coefficient is lower than that of small-world network but higher than that of random network. This is because unlike in small-world networks, there is no ordering in the connectivity amongst nearest neighbour nodes of a particular node in scale-free networks. However, since a scale-free
network is constructed by the property of a new node getting preferentially attached to nodes having higher degree, the clustering coefficient is higher than that of random network.

Next we analyzed the dependence of these graph properties on the network size, $N$, for the four topological networks. The results are summarized in Fig. 2.8. In regular networks, we expect the characteristic path length to increase linearly with the network size, $N$. This is clear from Fig. 2.8(a), where an exponential increase on the semi-log scale is observed. This increase is much faster in regular networks compared to small-world (Fig. 2.8(c)) and random networks (Fig. 2.8(e)). In random networks, due to random rewiring and absence of local clustering, the path length reduces drastically and the dependence is linear on the logarithmic scale. Thus, in case of small-world and random networks, average path length exhibits a slower logarithmic increase with the size of the network. In scale-free networks, the characteristic path length exhibits a logarithmic increase as shown in Fig. 2.8(g).

Since all the nodes have the same degree in the case of regular network, the average clustering coefficient is expected to be independent of the network size, as is clear from Fig. 2.8(b). Since small-world networks are also strongly clustered (except for a few long-range connections), the clustering coefficient is found to be independent of network size (Fig. 2.8(d)). The clustering coefficient of the random graphs is given by $\sim d/N$, where $d$ is the average degree and $N$ is the size of the network suggesting a linear decrease in the clustering coefficient with increase in network size, as observed in Fig. 2.8(f). In the scale-free networks, the clustering coefficient decreases with increase in network size (Fig. 2.8(h)).
Fig. 2.8: Characteristic path length and clustering coefficient as a function of network size, $N$, shown for various networks: (a), (b) Regular network, (c), (d) Small-world network, (e), (f) Random network, and (g), (h) Scale-free network.

### 2.3.2 Degree Distribution

In case of regular network, where each node in the 1000-node network has six nearest neighbour connections ($k = 6$), the degree distribution exhibits a delta function at $k = 6$ as shown in Fig. 2.9(a). As a result of a few long-range interactions, small-world network exhibits a spread about the average degree ($k = 6$) as shown in Fig. 2.9(b), for $p_r = 0.09$ (plotted for ten different
realizations). In Fig. 2.9(c) is shown the degree distribution for the random network. It follows Poisson distribution suggesting that on an average all nodes have similar connectivity with nodes having degree larger than average degree being very small, shown by the long tail on the right.

![Degree distribution graphs](image)

**Fig. 2.9:** Degree distribution, $k$ vs $P(k)$ (fraction of nodes having degree $k$) (plotted for ten realizations) (a) Regular, (b) Small-world and (c) Random networks.

The scale-free network used in this study exhibits a power-law degree distribution, $P(k) \sim k^{-\gamma}$ with $\gamma = 2.77$ as shown in Fig. 2.10.
2.3.3 Centrality Measures

Graph theory provides a rationale for choosing “critical” nodes in small-world and scale-free networks by an analysis of the graph properties. Importance of a node in a network based on the stability of the network in case of a targeted versus random attack and also in terms of information flow in the network is well studied. It has been observed that both small-world and scale-free networks contain few nodes with very high-degree, called “hubs”, because of their stabilizing effect on the network; targeted removal of hubs is known to collapse the network in a large number of networks such as World Wide Web. Nodes exhibiting high-betweenness or high-closeness values govern the information flow in the network and targeted removal of these nodes can also result in the collapse of the network (Newman et al., 2006). The topological properties of a network are well captured in the adjacency and Laplacian matrices. The spectrum of graph, $G$ is the set of eigen values of its adjacency or Laplacian matrix. A graph with $N$ nodes has $N$ eigen values $\lambda_j$, and it is useful to define its spectral density as

$$\rho(\lambda) = \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - \lambda_j)$$

which approaches a continuous function if $N \to \infty$. The interest in spectral properties is related to the fact that the spectral density can be directly linked to the graph’s topological features, since its $k^{th}$ moment can be written as
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\[ \frac{1}{N} \sum_{j=1}^{N} (\lambda_j)^k = \frac{1}{N} \sum_{i_1, i_2} A_{i_1, i_2} A_{i_2, i_3} \cdots A_{i_k, i_1} \]  

2.9

i.e., the number of paths returning from the same node in the graph. Note that these paths can contain nodes that were already visited. It has been studied that eigenvalue spectrum of these matrices provide information about structural properties of the networks they represent and also acts as a practical tool for the classification of networks in general (Barabasi and Albert, 2002). Another study has shown that the spectrum provides information about the contribution of each node in the network (Vishveshwara et al., 2002). Here, we discuss the various centrality measures used in our study to identify critical nodes in the network. The analysis of these centrality measures on the graph, \( G \) in Fig. 2.1 is summarized in Table 2.2.

**Degree:** It is one of the simplest measures of centrality and is given by Eqn. 2.2. Each term \( a_{ij} \) of the adjacency matrix takes value ‘1’ if node \( j \) is adjacent to node \( i \), else ‘0’. Thus, the degree of node \( i \) is the total number of its neighbours. It has been found that if nodes having high degree are targeted for removal from the network one by one, the average path length keeps increasing until the network completely collapses. For example, the high degree node in Fig. 2.1 is node 6 and its removal from the network will make the set of nodes (7-10) unreachable from the remaining nodes in the network.

**Betweenness:** For a given node \( i \), it is defined as the number of shortest paths (also called geodesics) between pairs of other vertices that traverse through it (Freeman, 1977). Thus, if \( g_i(st) \) is the number of geodesic paths from node \( s \) to node \( t \) that pass through \( i \), and if \( n_{st} \) is the total number of geodesic paths from \( s \) to \( t \), the betweenness of node \( i \) is given by

\[ b_i = \frac{1}{2N(N-1)} \sum_{s<t} g_i^{(st)} / n_{st} \]  

2.10

It is a measure of the influence of a node over the flow of information between other nodes in the network. These nodes can thus be regarded as central points controlling the communication among other nodes in the network. In Fig. 2.1, node 6 has the highest value of betweenness (see Table 2.2) since the maximum number of geodesic paths pass through it. It is critical to the flow of information from the nodes on the left (1-3) to the nodes on the right (7-10). Removal of this node will break the network into two clusters viz., (1-5) and (7-10). It may be noted that all the periphery nodes, i.e. 1, 2, 3, 7, 8, 9, and 10 have betweenness value ‘0’. This is because
through these nodes, no path exists between any pair of vertices in the graph. Nodes 4 and 5 also have high betweenness values since upon their removal, a number of geodesic paths will cease to exist.

**Closeness:** It is computed as the average of all the shortest paths between a node \(i\) and all other nodes in the network reachable from it (Newman, 2008) and is given by:

\[
CL_i = \frac{1}{(N_r - 1)} \sum_{j=1}^{N_r} L_{ij}
\]

where \(N_r\) is the number of nodes, \(j\) reachable from node \(i\) in the network. This measure computes the average connectivity of a node with the rest of the nodes in the network, that is, it defines how close in the topological space a node is to all the other nodes in the network. Thus, closeness can be regarded as a measure of how long it will take for information to spread from a given node to the other nodes in the network. Nodes 4, 5 and 6 in the graph in Fig. 2.1 have high values of closeness compared to the other nodes since these nodes are responsible for increasing the average connectivity between other nodes in the network. Based on closeness centrality, we can say that nodes 4, 5 and 6 are the most important nodes in the network. Nodes 4, 5, and 6 have very high closeness values since they are responsible for providing paths between pairs of nodes from the sets (1 - 3) and (7 - 10), hence decreasing the total path length between these pairs of nodes. If nodes 4, 5 or 6 were removed, no path between the set of nodes on the left (1 - 3) and the set of nodes on the right (7 - 10) would exist. Hence, nodes 4, 5 and 6 are critical to the flow of information in this network. Node 5 has the same closeness value as node 6, but its degree and betweenness value is much lower than that of node 6. Thus, though both are equidistant from all other nodes in the network, node 6 is more important that node 5. This suggests that one should analyse various centrality measures of a node to identify its importance in the network.

**Eigenvector centrality:** The importance of a node is not independent of the centrality of all other nodes connected to it, *i.e.*, the centrality of a node is higher if the nodes connected to it also have high centrality values. The eigenvalues and eigenvectors of matrices associated with a graph, *viz.*, adjacency matrix, Laplacian matrix, etc. provide information on the structure and topology of the graph and their analysis is called graph spectral analysis (Biggs, 1998). The magnitude of the vector components of the largest eigenvalue of these matrices correspond to the contribution of each node to the overall connectivity in the graph: the higher the magnitude,
more important the node in the graph (Vishveshwara et al., 2002). Thus, analysis of the spectra of the graph provides a measure for the importance of a node in a network. We denote the eigenvector corresponding to the principal (largest) eigenvalue of the adjacency matrix as $A_{lev}$ and that of the Laplacian matrix as $L_{lev}$. For the graph given in Fig. 2.1, both the eigenvector components, $A_{lev}$ and $L_{lev}$ have the highest magnitude for node 6. These eigenvector components also exhibit nodes having similar connectivity pattern in the network. For example, nodes 1 - 3 and 7 - 10 have a similar pattern in connectivity and hence similar values of eigenvector components.

Based on eigenvector centrality, node 6 is the most important node in the network. This is because it has the highest degree and is connected to the high betweenness and closeness node (5). Since nodes 1, 2 and 3 are present in a similar environment, i.e., they are connected to node 4 only, these nodes have equal values for all centrality measures. Similarly, nodes 7, 8, 9 and 10 have the similar values for all the centrality measures. Nodes 7, 8, 9, and 10 have a higher value in magnitude of eigenvector centrality compared to Nodes 1, 2, and 3. This is because these nodes are connected to a higher degree node (6) compared to node 4. Thus, we observe that centrality measures in addition to identifying critical nodes also help in identifying clusters in the network. It may be noted that there is an overlap of nodes identified by various high-centrality nodes. Thus, a node which has high-centrality value for various measures is definitely an

Table 2.2: Values of five centrality measures, viz, degree ($k_i$), betweenness ($b_i$), closeness ($Cl_i$), eigenvector centrality for adjacency matrix ($A_{lev}$) and the eigenvector centrality for Laplacian matrix ($L_{lev}$) for graph in Fig 2.1

<table>
<thead>
<tr>
<th>Node</th>
<th>$k_i$</th>
<th>$b_i$</th>
<th>$Cl_i$</th>
<th>$A_{lev}$</th>
<th>$L_{lev}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0.35</td>
<td>-0.1568</td>
<td>0.0342</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>0</td>
<td>0.35</td>
<td>-0.1568</td>
<td>0.0342</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>0</td>
<td>0.35</td>
<td>-0.1568</td>
<td>0.0342</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.58</td>
<td>0.50</td>
<td>-0.3717</td>
<td>-0.1737</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.56</td>
<td>0.56</td>
<td>-0.4106</td>
<td>0.2588</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
<td>0.72</td>
<td>0.56</td>
<td>-0.6015</td>
<td>-0.8824</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0</td>
<td>0.38</td>
<td>-0.2538</td>
<td>0.1737</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0.38</td>
<td>-0.2538</td>
<td>0.1737</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0.38</td>
<td>-0.2538</td>
<td>0.1737</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>0</td>
<td>0.38</td>
<td>-0.2538</td>
<td>0.1737</td>
</tr>
</tbody>
</table>
important node. However, depending on the properties one is investigating, different centrality measures need to be analysed.

### 2.4 Modeling Dynamics on a Network

Most studies on dynamical networks have focused on the underlying network topology changing in time. For example, the World Wide Web is constantly growing in size in terms of its number of nodes while the food web is diminishing. In many physical and biological systems, one encounters a situation where the nodes are dynamical entities evolving in time and the connection topology may remain fixed or change in time.

Based on these properties, dynamical networks can be classified into three types. Networks having their

1. Topology evolving in time,
2. Topology fixed, but dynamics on nodes evolving in time,
3. Topology and dynamics on individual nodes both evolving in time.

Thus in the analysis of dynamical networks, the emphasis could be either on the interplay between the local and global dynamics or on the dynamically evolving topology. Typically, a combination of evolving topology and dynamics on individual nodes is the most interesting situation. It involves a coupling between two different time scales; the individual dynamics evolves on a faster time scale and the network responds to that dynamics and changes/evolves in topology at a slower time scale. In neural networks, one has a fast activity of the dynamics of nodes (neurons) and a slow learning dynamics that changes the weights of the connections, called synapses, in response to the activity correlation between neurons. But in order to study this combination, we need to first understand them individually. Hence, in our study, we investigate the role of topology in the dynamics of the network.

Complex nonlinear dynamics of extended systems have been extensively studied using coupled map lattices with nearest neighbour interactions. In real systems, one is likely to observe very few systems having such a strictly regular arrangement of individual entities. Many real-world systems have been found to have random, non-local connectivity between their nodes. This randomness has been shown to drastically affect the geometrical properties of the networks with non-regular topology. For instance, characteristic path length of a small-world network is significantly lower than that of a regular network. While the effect of the underlying topology of the network on its geometrical properties has been studied extensively, their impact on the
dynamics is still unclear. Hence, in our study we aim to analyze the change in dynamics with the change in heterogeneity in the spatial coupling of the nodes in the network. Here, we have considered two situations, (1) when the disorder in the spatial coupling is low, and (2) when the spatial coupling is highly heterogeneous, but not random. The first situation is modeled by a small-world connection topology while the second situation is modeled by a scale-free topology.

Here we analyze a coupled system of dynamical equations on different underlying topologies, and the emphasis is on manipulating/regulating the resulting global dynamics of the system emerging from the interactions between the local dynamics of the individual elements (Jost, 2005). Analysis of the dynamics on spatially extended systems is extensively studied on coupled map lattice models wherein the connection topology is nearest-neighbour diffusive coupling. Here, the dynamical system is discretized on time and space but the continuous state variables are preserved. In this thesis we present our analysis when the dynamical processes on each node is governed by (1) Logistic map, a one-dimensional nonlinear map (May, 1976), and (b) neuronal map, a two-dimensional nonlinear map (Chialvo, 1995). The dynamical behaviour of the individual maps is briefly discussed below.

### 2.4.1 Logistic map

Logistic map is a recurrence relation of degree 2, very commonly used to describe complex behaviour arising from simple non-linear dynamical equations. The map was popularized in a seminal 1976 paper by Robert May (1976). Mathematically, it is given by

\[ f(x) = rx(1 - x) \quad 0 < r \leq 4, \quad 0 \leq x \leq 1 \]  

and exhibits rich dynamical behaviour as a function of \( r \). The temporal behaviour of the logistic map is shown in Fig. 2.11. It exhibits period-doubling route to chaos with increasing \( r \) with a wide variety of dynamical behaviour ranging from periodic, quasi-periodic, intermittency to fully developed chaos.
Analysis of underlying topology on dynamical networks

![Fig. 2.11](image): Period-doubling route to chaos exhibited by a logistic map as a function of $r$ shown for a representative node in a regular network with $\epsilon = 0.5$. 200 time steps have been plotted after removing the transients.

The spatiotemporal dynamics of the network is governed by the equation (Jost, 2005):

$$x(i, n + 1) = f(x(i, n)) + (\epsilon / k_i) \sum_{j, j \sim i} [f(x(j, n)) - f(x(i, n))] = F_i$$  

where $n$ is the time step, $i = 1, 2, \ldots, N$, the nodes in the network, $\epsilon = [0,1]$ is the coupling strength, $k_i$ is the number of neighbours of node $i$ (i.e., nodes connected to node $i$) and $j \sim i$ corresponds to edge between nodes $j$ and $i$, where $j = 1, 2, \ldots, k_i$.

### 2.4.2 Neuron map

It is known that activity in neuronal networks contains two distinct timescales (i) a fast timescale characterized by repetitive spiking; and (ii) a slow timescale with bursting activity, where neuron activity alternates between a quiescent state and spiking trains (Belykh et al, 2005). A characteristic feature of cortical circuits is that they can produce common rhythmic bursting, while its individual neurons, when isolated, show irregular bursts (Coombes and Owen, 2005). Many mathematical models emulate this spiking–bursting behaviour, ranging from differential equations to discrete-time maps. A number of models such as the Hodgkin–Huxley (1952), Fitzhugh–Nagumo (1962) or Hindmarch–Rose differential equations (1984) have been used in past to model neuronal network. Owing to its simplicity, we have used a two-dimensional map proposed by Chialvo to model neuron dynamics (Chialvo, 1995).

A neuron is an excitable system. In general, the dynamics of an excitable system can be described by two state variables termed the “potential” and the “recovery” variables (Chialvo, 1995):
Analysis of underlying topology on dynamical networks

\[
x_{n+1} = f_1(x_n, y_n) = x_n^2 \exp(y_n - x_n) + k \\
y_{n+1} = f_2(x_n, y_n) = ay_n - bx_n + c
\]

where \(x\) and \(y\) are the state variables related to neuron membrane potential and recovery current. The subscript \(n\) refers to the iteration step. The model has four parameters: Parameter, \(a\) determines the time constant of reactivation, \(b\) determines the rate of inactivation and \(c\) determines the maximum amplitude of the recovery current. The parameter \(k\) may be a constant bias or a time-dependent external stimulation. It determines the complexity of dynamics of the system. The parameters in Eqn. 2.14 are chosen so as to make the dynamics completely chaotic as evident from Fig. 2.12 (\(a = 0.89, b = 0.18, c = 0.28, k = 0.03\)). The bifurcation behaviour of the \(x\) variable w.r.t parameter \(k\) is shown in Fig. 2.12(a). Below \(k\sim 0.02\), the system is stable while around \(k = 0.03\) region, the system dynamics is completely chaotic. In the region around \(k = 0.048\), the dynamics changes from chaotic to highly-periodic. This is shown in Fig. 2.12(b). In Fig. 2.13, the phase plots are plotted for two parameter regimes: (1) chaotic (\(k = 0.03\)) and (2) highly periodic (\(k = 0.048\)). In Fig. 2.14(a) and Fig. 2.14(b) the temporal behaviour of the neuron map is shown for variables \(x\) and \(y\) respectively. Here, the basic rhythm is composed of sets of initially relatively large oscillations, followed by one or more rapid secondary oscillations increasing in amplitude. The interval between large spikes as well as the number of small oscillations are both variable. Fig. 2.15 shows the temporal behaviour for \(x\) and \(y\) variables for \(k = 0.048\). Here all the spikes are equal in amplitude without any intermediary smaller spikes. This indicates that the dynamics at \(k = 0.03\) is highly chaotic while at \(k = 0.048\), it is periodic. It has been found that high frequency, low amplitude oscillations such as the ones in Fig. 2.15 occur during arousal and attention while low-frequency, high amplitude oscillations activity such as those in Fig. 2.14 occurs during the quiescent state (Thompson, 2000).
Fig. 2.12: The bifurcation diagram of the $x$ variable with $k$ varying from (a) 0.0 to 0.08, and (b) 0.045 to 0.051 plotted for 200 time steps.
Analysis of underlying topology on dynamical networks

Fig. 2.13: Phase plots for two regimes (a) $k = 0.03$ (Chaotic) and (b) $k = 0.048$ (Periodic)

Fig. 2.14: The temporal behavior of (a) $x$ variable, (b) $y$ variable in the chaotic regime ($k = 0.03$) for initial conditions of $x = 0, y = 0$ shown.
Analysis of underlying topology on dynamical networks

Fig. 2.15: The temporal behavior of (a) x variable, (b) y variable in periodic regime \((k = 0.048)\) for initial conditions \(x = 0, y = 0\) shown.

We know that brain is a very complex structure comprised of neurons. In order to study the collective dynamics of neurons, we model it as a coupled map system. The local dynamics in this case is modelled by Chialvo’s neuron map. The spatiotemporal dynamics of such a neuron network is governed by the equation:

\[
x(i,n+1) = (1 - \varepsilon) f_1(x(i,n), y(i,n)) + (\varepsilon / k_i) \sum_{j \sim i} [g(x(j,n))] \equiv F_i^1
\]

\[
y(i,n+1) = f_2(x(i,n), y(i,n)) \equiv F_i^2
\]

where \(f_1\) and \(f_2\) are as defined in Eqn. 2.14, \(n\) is the time step, \(i = 1, 2, \ldots, N\), the nodes in the network, \(\varepsilon = [0,1]\) is the coupling strength, \(k_i\) is the number of neighbours of node \(i\) (i.e., nodes connected to node \(i\)) and \(j \sim i\) corresponds to edge between nodes \(j\) and \(i\), where \(j = 1, 2, \ldots, k_i\). The function \(g(x)\) can take various forms; here for simplicity, we consider, \(g(x) = x\).
2.5 Control of Spatiotemporal Chaos

Many physical and biological systems exhibit a variety of spatiotemporal dynamics, from stable to chaotic that can change under pathological conditions and impair their normal functions. The connection topology in these systems need not always be nearest-neighbour diffusive coupling but may exhibit heterogeneity in the spatial coupling, resulting in long-range and/or densely coupled nodes. Thus being able to control the altered dynamics of spatiotemporal systems on non-regular topology has potential for wide ranging applications in real and artificial systems, such as neuronal networks, metabolic networks, food webs, the networks of feeding relationships among the species within habitats, etc. Over the past few years, many different approaches have been proposed for the control of the chaotic dynamics. Most of the methods either require a priori knowledge of the system dynamics, such as, the stable or unstable fixed points and periodic orbits; or, involve direct modification or tracking of the system parameters. The approach used in the current study which involves the application of a constant perturbation or pinning signal in the spatial domain requires no a priori information about the system dynamics. The sign and the strength of the pinning signal alone determine the control of spatiotemporal dynamics to the desired states. In an earlier study on applying a constant pinning signal to a simple mathematical equation representing population growth in ecology, it was shown that the suppression of chaos and the initiation of a reversal of period-doubling eventually followed by period halving could be achieved (Sinha and Parthasarathy, 1996). It was suggested that this type of phenomenon occurs in models of the crow-of-thorns star fish, insect population, perennial grass inhibited by plant litter, host-parasitoid interactions and the systems of competing species (Stone, 1993). Here we propose to use the approach for controlling spatiotemporal dynamics by applying an external pinning as follows (Parekh et. al., 1998):

\[ x(i,n+1) = f(x(i,n)) + \frac{\epsilon}{k_i} \sum_{j, j \neq i} [f(x(j,n)) - f(x(i,n))] + p_s(i,n) \equiv F_i + p_s(i,n) \quad 2.16 \]

where \( p_s(i,n) \) represents the pinning strength at the \( i^{th} \) site at the \( n^{th} \) time step. \( F_i \) is the CML for a logistic map as given in Eqn. 2.13.

For the neuron map, the pinning signal is applied only to the \( x \) variable as shown below:

\[ x(i,n+1) = (1 - \epsilon) f_1(x(i,n), y(i,n)) + \frac{\epsilon}{k_i} \sum_{j, j \neq i} [g(x(j,n))] + p_s(i,n) \equiv F_i^1 + p_s(i,n) \]
\[ y(i,n+1) = f_2(x(i,n), y(i,n)) \equiv F_i^2 \quad 2.17 \]
The pinning term can assume negative or positive values depending on the nature of the local dynamics. It has been shown by Parekh and Sinha (2002) that for the logistic map, the negative pinning values result in suppression of chaos in the dynamics while positive values enhance/induce chaos. Here we have analyzed the efficacy of this approach on dynamical networks with non-local connection topologies, viz., small-world, scale-free and random.

The Lyapunov characteristic exponent of a dynamical system is a quantity that characterizes the rate of separation of infinitesimally close trajectories. That is, it quantifies the growth rate of trajectories for infinitesimally small perturbations. The rate of separation can be different for different orientations of initial separation vector. Thus, there is a **spectrum of Lyapunov exponents**— equal in number to the dimensionality of the *phase space*. The largest Lyapunov exponent is called the **Maximal Lyapunov exponent** (MLE), and it takes a positive value if the system dynamics is chaotic and negative values for periodic and stable behaviour. Thus it is a useful measure for characterizing the complexity of the dynamics. The Lyapunov exponent (LE) for a one dimensional map is given by:

$$\lambda_{\text{max}} = \lim_{t \to \infty} \frac{1}{t} \sum_{n=1}^{t} \log \left( \frac{d}{dx} f(x(n)) \right)$$

where $f(x(n))$ is the function at time step, $n$. LE is calculated by averaging the divergence over $t$ time steps.

The characterization of the spatiotemporal dynamics in terms of the Lyapunov exponents is discussed below. For a spatially discretized system with $n$ variables on a network of size $N$, we have ‘$nN$’ dependent variables which can be represented by the Jacobian matrix of size ‘$nN \times nN$’. For the logistic map (single variable system) on a network of size $N$, the Jacobian is:

$$J = \begin{bmatrix}
\frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \ldots & \frac{\partial F_1}{\partial x_N} \\
\frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \ldots & \frac{\partial F_2}{\partial x_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial F_N}{\partial x_1} & \frac{\partial F_N}{\partial x_2} & \ldots & \frac{\partial F_N}{\partial x_N}
\end{bmatrix}$$

where $F_i$ correspond to the coupled logistic map on each node $i$. For the neuron map, the Jacobian is:
where $F_i^1$ and $F_i^2$ define the 2-dimensional coupled neuron map on each node $i$. In order to take care of the divergence of two nearby initial conditions, Gram-Schmidt orthonormalization (Wolf et al., 1985) is applied to compute the maximum Lyapunov exponent.

In the next chapter, we present our results on these two coupled map models for different connection topologies.
3 Results And Discussion

In Chapter 2, we introduced the concept of graph theory and its applications. We also discussed how graph theory can be used to model real-world systems. The construction of four topological networks, *viz.*, regular, small-world, random and scale-free, by Watts-Strogatz Model and Barabasi-Albert algorithm was described. The mathematical forms of various graph properties and their significance in characterizing the four topological graphs and identifying critical nodes in them was discussed. We then introduced the concept of spatiotemporal dynamics and the significance of modeling it on real-world systems. The mathematical forms of two maps, *viz.*, logistic map and neuron map, used to model dynamics on the four topological networks, were discussed. They have been chosen for our study since both these maps are known to exhibit rich dynamical behavior. The concept of controlling dynamics on these maps by applying external perturbation or *pinning* was introduced. In this chapter, we first present the spatiotemporal dynamical behavior for the two maps (logistic map and neuron map) on different topological networks. We then analyze the role of the underlying topology of the network in achieving control of spatiotemporal dynamics by pinning.

For the numerical simulations of the logistic map, the network size is $N = 1000$ and the initial conditions for the variable $x$ in the logistic map were randomly chosen in the range $0.5 \pm 0.01$. For control studies, first 500 transients were discarded before applying pinning. For the logistic map, the analysis has been carried out for two cases: (i) strong chaos ($r = 3.9$) and (ii) weak chaos ($r = 3.6$). The dynamics of a representative node of a network, where each node has been modelled as a logistic map is shown in Figure 3.1. For our analysis in the strong chaos regime ($r = 3.9$), coupling strength chosen is $\epsilon = 0.5$ while in the low chaos regime ($r = 3.6$), coupling strength chosen is $\epsilon = 0.25$ (Fig 3.3). It is important to note that the iterates of the logistic map diverge under strong negative pinning strength values. This problem is overcome by applying pinning only if $x \geq |p_s|$ (Parekh et al., 2002).

In case of the neuron map, the initial conditions are $x = 0$, $y = 0$. First 1000 transients were discarded before pinning. The analysis has been carried out for two regimes (i) chaotic ($k = 0.03$) and (ii) periodic ($k = 0.048$). A very low coupling strength, $\epsilon = 0.05$ has been chosen for the analysis in this case.
In section 3.1, we present our results for the case where dynamics on each node is modeled by a logistic map. In section 3.2, results are presented for the case where the dynamics on each node is modeled by a neuron map.

### 3.1 Logistic Map

The spatiotemporal dynamics of a coupled map lattice is governed by the Eqn. 2.10 presented in the previous chapter. In order to investigate the temporal behaviour of the system, the dynamics of a representative node in the network is plotted for 200 time steps as a function of $r$. This is shown in Fig. 3.1.

![Fig. 3.1](image)

**Fig. 3.1:** Period-doubling route to chaos as a function of $r$ shown for a representative node with $\varepsilon = 0.5$ for (a) regular, (b) small-world, (c) scale-free, and (d) random network, where the dynamics on each node is modeled by a logistic map. 200 time steps have been plotted after removing the transients.

It is clear from the figure that the four network topologies exhibit similar temporal behavior. A period doubling route to chaos is exhibited in all the four cases similar to a single logistic map. This suggests that the topology does not have any significant effect on the asymptotic dynamical behaviour of a node in the network.
Since the four topological networks are different from each other in terms of their connectivity patterns, we expect the strength of coupling to play some role in the dynamics of the four networks. In an earlier study on synchronization of network dynamics, Jost et al (2005) investigated the role of coupling $\varepsilon$, in the synchronization of network dynamics. They showed that the networks having non-regular topology tend to synchronize over a wider range of parameters. They investigated the degree of synchronization of the network by defining a function

$$\sigma^2 = \sum_{i=1}^{N} (x_i(t) - \bar{x}(t))^2$$  \hspace{1cm} \text{(3.1)}

where $N$ is the size of the network, $t$ is the time step, $i$ represents a node in the network and $\bar{x}(t) = (1/N) \sum_{i=1}^{N} x_i(t)$. The dynamics on each of the nodes in the network is said to be synchronized if $\sigma^2(t) \to 0$ as $t \to \infty$. Figures 3.2 and 3.3 show the synchronization behavior of the four topological networks in the high chaos ($r = 3.9$) and low chaos ($r = 3.6$) regimes respectively. We observe that in the high chaos regime, scale free networks synchronize for $\varepsilon > 0.64$ while the other three networks, viz., regular, small-world and random do not synchronize at all (Fig. 3.2). However, in the low chaos regime, both scale free and random networks synchronize over a wider range of coupling strength as compared to regular and small-world networks. Scale free networks synchronize for $\varepsilon > 0.28$ and random networks synchronize for $\varepsilon > 0.58$ (Fig. 3.3). Thus, we observe that dynamics on networks having non-regular topology synchronize over wider range of coupling strength compared to networks having primarily nearest-neighbour connections. This indicates the role of connection topology on the synchronizability of the coupled spatiotemporal dynamics.
As discussed in Chapter 2, the importance of controlling spatiotemporal dynamics is evident. Having identified the parameter range for which the global network dynamics is spatiotemporally chaotic, we now discuss below their control by applying an external perturbation on:

- all nodes uniformly
• nodes chosen at regular intervals
• nodes having high values of centrality measures, and
• nodes randomly chosen

In real, practical situations, it has been found that the spatiotemporal chaos is low. Hence we will first present our results for pinning in the low chaos regime and then for the high chaos regime.

3.1.1 Uniform Pinning

The most trivial case of pinning is to uniformly pin all the nodes in the network with a constant pinning strength, i.e. \( p_s(i,n) = p_s \). In an earlier work, it has been shown that the negative pinning values result in suppressing chaos in the logistic map while positive values enhance/induce chaos (Parekh et al, 1998). In Fig. 3.4 and 3.5 is shown the bifurcation diagram w.r.t. pinning strength on the four topological networks operating in low and high chaos regimes respectively. The pinning strengths required to control the dynamics are the same across the four topological networks.

Fig. 3.4: Temporal behaviour for a representative node shown when all nodes are uniformly pinned with strength \( p_s \), in (a) Regular, (b) Small-world, (c) Scale-free, (d) Random networks for \( r = 3.6 \) and \( \varepsilon = 0.25 \).
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Figure 3.4 shows the temporal behavior of a representative node in the four networks when the global network dynamics is weakly chaotic ($r = 3.6$) and $\varepsilon = 0.25$. Very low pinning strength (-0.02) suffices in achieving control of spatiotemporal dynamics. In figure 3.5 is shown the dynamics on a representative node in the four networks when the network dynamics is strongly chaotic ($r = 3.9$, $\varepsilon = 0.5$). In this case much higher pinning strength (-0.13) is required to suppress chaos and the system exhibits a 4p state. Similarly, by uniformly pinning all nodes with a pinning strength, $p_s = -0.16$, we can achieve a two-period state. A fixed point state can be achieved by applying pinning strength, $p_s \geq -0.29$. We observe that by pinning all nodes in the network, the temporal dynamics for a representative node against the pinning strength, $p_s$, for the four networks exhibit no real differences. Table 3.1 shows the minimum values of pinning strength required to control the spatiotemporal dynamics in the four networks to any periodic state, when all the nodes in the network are pinned.

Table 3.1: Pinning strength, $p_s$, required to control spatiotemporal chaos (STC) to a four-period state in the four topological networks, when all nodes in the four networks are pinned, i.e., $p_d = 100\%$

<table>
<thead>
<tr>
<th>$r$</th>
<th>Reg</th>
<th>SW</th>
<th>SF</th>
<th>Rand</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.9</td>
<td>-0.13</td>
<td>-0.12</td>
<td>-0.13</td>
<td>-0.13</td>
</tr>
<tr>
<td>3.6</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.02</td>
<td>-0.02</td>
</tr>
</tbody>
</table>
3.1.2 Non-Uniform Pinning

In real practical situations it may not be efficient and feasible to control probes over the entire spatial domain. Instead, it would be beneficial to apply external perturbation at a lower pinning density. In order to choose a set of minimal number of probes for pinning, three different pinning strategies are considered here. Here we propose the use of geometrical properties of the underlying connection topology for selecting nodes to exert pinning. Thus in the case of regular and small-world networks, wherein the connectivity pattern is predominantly nearest-neighbour, we consider pinning nodes chosen at regularly spaced intervals. In the case of small-world and scale-free networks we consider pinning nodes based on their centrality values. Finally for all the four networks, the simplest proposition of choosing nodes randomly for pinning is considered. The controllability of the network dynamics under various pinning situations is then assessed.

3.1.2.1 Pinning regularly spaced nodes: Exploiting the inherent regularity in the connection pattern of regular and small-world networks, we now analyse the controllability in these networks on exerting pinning at regular intervals, say, every 2\textsuperscript{nd}, 3\textsuperscript{rd}, 4\textsuperscript{th}, …, nodes. In the case of random and scale-free networks, no such ordering of the nodes is possible, and hence have not been analysed in this case. The minimum strength of pinning required in suppressing chaos at varying pinning densities is summarized in Table 3.2.

<table>
<thead>
<tr>
<th>$p_d$</th>
<th>$r = 3.6$</th>
<th>$r = 3.9$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reg</td>
<td>SW</td>
<td>Reg</td>
</tr>
<tr>
<td>100%</td>
<td>-0.02</td>
<td>-0.13</td>
</tr>
<tr>
<td>50%</td>
<td>-0.02</td>
<td>-0.04</td>
</tr>
<tr>
<td>20%</td>
<td>-0.06</td>
<td>NC</td>
</tr>
</tbody>
</table>

It may be noted that low pinning strengths and pinning densities suffice in suppressing spatiotemporal chaos on both the networks in the weak chaotic regime ($r = 3.6$). For example, a very low pinning value ($\sim 0.06$) suffices in suppressing spatiotemporal chaos (STC) on both regular and small-world topology on pinning every 5\textsuperscript{th} node (i.e., at 20\% pinning density). This is shown in Fig. 3.6. In the highly-chaotic regime, much higher pinning strengths along with higher pinning density i.e., every alternate node pined is required to suppress STC. This is depicted in Fig. 3.7.
Fig. 3.6: Temporal behavior of a representative node w.r.t. pinning strength, $p_s$, is shown on pinning every fifth node in (a) Regular, and (b) Small-world networks for $r = 3.6$ and $\epsilon = 0.25$

Fig. 3.7: Temporal behavior of a representative node w.r.t. pinning strength, $p_s$, is shown on pinning every second node in (a) Regular, and (b) Small-world networks for $r = 3.9$ and $\epsilon = 0.5$

From Fig. 3.6 and 3.7, we observe that no significant difference in the strength or density of pinning is required for controlling chaos on a small-world network compared to a regular network. This is contrary to our expectation, i.e., no effect of the long-range coupling in the small-world network is observed in controlling the dynamics. This may be because of very small percentage of nodes exhibiting long-range coupling (~ 263 out of a total of 3000 edges are long-range) which only acts as noise in the presence of strongly coupled nearest-neighbour interactions in both the networks (Jampa et al., 2007).

3.1.2.2 Pinning High-Centrality Nodes: Graph theory provides a rationale for choosing “critical” nodes in small-world and scale-free networks by an analysis of the topological
properties of the graph. Importance of a node in a network based on the stability of the network in case of a targeted versus random attack and also in terms of information flow in the network is well studied. It has been observed that both small-world and scale-free networks contain few nodes with very high-degree, called “hubs”, because of their stabilizing effect on the network; targeted removal of hubs is known to collapse the network. Nodes exhibiting high-betweenness or high-closeness values, govern the information flow in the network. Since the topological properties of a network are well captured by the adjacency and Laplacian matrices, an analysis of the eigen spectra of these matrices also provide the contribution of each node in the network (Vishveshwara et al., 2002). It has been shown that targeted removal of these “critical” nodes (that is nodes having high values of these centrality measures) can result in the collapse of the network (Newman, 2006). The effect of the targeted removal of various high-centrality is assessed by computing the global efficiency defined by Eqn 2.7. The reduction in the global efficiency of the networks on gradual removal of high-centrality nodes is depicted in Fig 3.8(a) and 3.8(b) for small-world and scale-free network respectively.

**Fig. 3.8:** Global Efficiency ($E_{global}$) plotted as a function of the percentage of removal of high centrality nodes from (a) small-world network and (b) scale-free network. The figure legend shows the symbols used to represent nodes having high value of a particular centrality measure.

It is clear from Fig. 3.8(a) that in the case of small-world network, targeted removal of nodes at regular intervals results in a much faster fall in the efficiency of the network followed by pinning high-degree and high-betweenness nodes. The efficiency is seen to fall more than
50% on targeted removal of ~ 20% of these nodes. The effect is not so prominent on removing the remaining high-centrality nodes, suggesting that pinning at regular intervals may be more promising in achieving control of the dynamics on a small-world topology. In the case of scale-free network, we observe in Fig. 3.8(b) that targeted removal of ~ 10% of high-betweenness or high-closeness nodes results in reducing the efficiency of the network by 50%, suggesting their possible influence in achieving control of the network dynamics. Similar reduction in efficiency on targeted removal of the remaining centrality measures requires ~ 15 – 20% of the nodes to be removed. In contrast to removing nodes based on their centrality measure values, on randomly removing nodes from the network, we observe that the decrease in global efficiency is not much. This is in agreement with the general observation that scale-free networks are more vulnerable to targeted attacks as compared to random attacks. Below we present our results on pinning “critical” nodes identified based on their centrality values and analyze the achievability of control in small-world and scale-free networks.

3.1.2.2.1 Small-World Network

Ten realizations of the small-world network have been constructed with the same rewiring probability, $p_r = 0.09$, for the analysis. The results of pinning various high-centrality nodes in a small-world network are summarized in Table 3.3. For different realizations of the network, the pinning strengths reported in the table are found to vary by ±0.04. The pinning densities reported are the minimum density of pinning sites required to achieve control. No global control was observed below these pinning densities. We observe that by pinning high degree and high betweenness nodes we can achieve control of STC. Suppression of chaos by pinning high-degree nodes

Table 3.3: Pinning strength and pinning density showing control of the dynamics on small-world topology by pinning nodes having high values of centrality measures, viz., degree ($k$), betweenness ($b$), closeness ($cl$), eigenvector corresponding to the largest eigenvalue of the adjacency matrix ($A_{lev}$), and eigenvector corresponding to the largest eigenvalue for a Laplacian matrix ($L_{lev}$).
nodes is observed only in the low-chaotic regime for minimum density of pinning 50% at \( p_s = -0.37 \) while a much higher pinning strength is required for suppressing chaos on pinning high-betweenness nodes. No control was observed on pinning nodes with high-closeness values or on choosing nodes based on the magnitude of the eigenvector components of adjacency and Laplacian matrices. In the case of the system exhibiting highly chaotic dynamics, control is not attainable by pinning any of the high-centrality nodes. When we compare with our results on regularly-spaced pinning, we observe that a very low pinning strength suffices even at 20% pinning density to control the dynamics. This is consistent with our results of global efficiency in Fig. 3.8(a), suggesting that global efficiency is an important measure in assessing not only the stability of the network, but also in identifying “critical” nodes for applying external perturbation for achieving global control of the network dynamics. Thus, we conclude that no significant role of long-range interactions is observed in controlling the chaotic dynamics on the small-world topology as no advantage on pinning graph centrality measures is observed compared to regularly-spaced pinning. This is not surprising since there is a strong inherent order in the arrangement of the nodes in the small-world network as a consequence of nearest neighbor interactions dominating in the network. The long range connections only seem to act as spatial noise in the coupling. Thus, we conclude that in the small-world network, control is easily attainable for very low pinning strengths and pinning densities in case of regularly-spaced controllers/probes similar to regular network. Figure 3.9 is shown the spatiotemporal behaviour on a small-world network at \( r = 3.6 \) when different pinning strategies are used, namely, at regular intervals, high-centrality nodes and at random.
In Figure 3.9(a) the spatiotemporal dynamics is weakly chaotic as indicated by the maximal Lyapunov exponent when no pinning is applied. On pinning every 5th node in a small-world network (Fig. 3.9(b)), global control of the network dynamics is observed ($\lambda_{\text{max}} = -0.04$) for very low pinning strength, $p_s = -0.06$. However a much higher pinning strength ($p_s = -0.37$) at high pinning density ($p_d = 50\%$) is required to achieve global control on pinning high degree nodes as shown in Fig. 3.9(c). When knowledge of the connection topology is taken into consideration, i.e. the nodes are randomly chosen for pinning, then a high pinning density ($p_d \sim 50\%$) is required to achieve control. The pinning strength required is very low ($p_s \sim -0.05$) compared to pinning high degree nodes. However, on choosing nodes randomly for pinning, global control is not always guaranteed. For ten different random configurations of nodes chosen for pinning, we observed control on only five occasions.

To further investigate the role of long-range interactions in controlling dynamics in small-world networks, another small-world network was constructed for probability of rewiring, $p_r = 0.01$ (much lower than $p_r = 0.09$, taken earlier for the analysis). This network has very few nodes (3.3\%) having degree greater than the average compared to small-world network constructed at $p_r = 0.09$ (~18.8\% nodes). It was found that similar pinning strengths are required for suppressing chaos in both the cases. In Table 3.4 is shown the pinning strength required to control STC in case of two different small-world networks.
Table 3.4: Comparison of pinning high centrality nodes on the small world networks constructed with rewiring probability 0.01 and 0.09 (r=3.6).

<table>
<thead>
<tr>
<th>Centrality measures</th>
<th>( p_d )</th>
<th>( p_r = 0.01 )</th>
<th>( p_r = 0.09 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Regular</strong></td>
<td>20%</td>
<td>-0.06</td>
<td>-0.06</td>
</tr>
<tr>
<td><strong>Random</strong></td>
<td>50%</td>
<td>-0.05</td>
<td>-0.05</td>
</tr>
<tr>
<td>( k )</td>
<td>50%</td>
<td>-0.35</td>
<td>-0.37</td>
</tr>
<tr>
<td>( b )</td>
<td>50%</td>
<td>-0.43</td>
<td>-0.42</td>
</tr>
<tr>
<td>Cl</td>
<td>50%</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td>( A_{lev} )</td>
<td>50%</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td>( L_{lev} )</td>
<td>50%</td>
<td>NC</td>
<td>NC</td>
</tr>
</tbody>
</table>

It is clear from the table that there is no significant difference in the values of pinning required to attain control of STC on varying the rewiring probability. That is no significant effect of increasing the long-range interactions is observed since the nearest-neighbour interactions predominate in governing the dynamics. The strength and density of pinning required to attain control of STC when nodes are spaced regularly, chosen randomly and chosen based on their centrality measure values are the same for different configurations of small-world network.

3.1.2.2.2 Scale-free network:

In this section we present our analysis of controlling the dynamics in a scale-free network in both weak and strong chaos regimes. We observe that suppression of spatiotemporal chaos is easily attainable in this case by pinning high centrality nodes. The pinning density and pinning strength required to achieve global control of STC by pinning various high centrality measure nodes have been summarized in Table 3.5 (results averaged over 10 realizations).

Table 3.5: Comparison of strength and density of pinning required for global control of spatiotemporal dynamics on scale-free network on choosing various high-centrality nodes. (\( p_s \pm 0.04 \) observed for ten realizations of scale-free network)

<table>
<thead>
<tr>
<th>Centrality measures</th>
<th>( r = 3.6 )</th>
<th>( r = 3.9 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k )</td>
<td>15% -0.30</td>
<td>50% NC</td>
</tr>
<tr>
<td>( b )</td>
<td>5% -0.22</td>
<td>25% -0.32</td>
</tr>
<tr>
<td>Cl</td>
<td>5% -0.22</td>
<td>25% -0.38</td>
</tr>
<tr>
<td>( A_{lev} )</td>
<td>10% -0.28</td>
<td>50% -0.44</td>
</tr>
<tr>
<td>( L_{lev} )</td>
<td>10% -0.25</td>
<td>50% -0.35</td>
</tr>
</tbody>
</table>

It may be noted that pinning as few as 5% of high-betweenness or high-closeness nodes suffice in suppressing weak chaos; increasing the pinning densities to 10 - 15%, global control is
achievable at much lower pinning strengths ($p_s \sim 0.16$). Control by pinning other high-centrality nodes is also achievable in this case, however at higher density and strength of pinning. There is an overlap of 88% of nodes among top 5% high betweenness and high closeness nodes. Hence similar pinning strengths are required to suppress chaos on choosing high betweenness or high closeness nodes. In the case of networks exhibiting strongly chaotic dynamics, control is achieved only on pinning high-betweenness or high-closeness nodes; for other centrality nodes, pinning strength required is very high even at 50% pinning density to be of any practical use. We observe that contrary to the expectation of targeted control of “hubs”, about 15% of high-degree nodes are required to be pinned to control weak chaos while no control is observed in the case of strong chaos. There is an overlap of 79% of the nodes among top 15% high degree and high betweenness nodes. This indicates that in addition to capturing high degree nodes, betweenness also captures nodes that connect different local regions in the network and hence pinning these nodes results in the global control of the network dynamics. The better performance observed on pinning the centrality measures, betweenness, closeness and eigenvector centrality compared to high-degree is because these identify the importance of node in the network not just by its connectivity (degree), but also depend on the connectivity of its neighbours. That is, a node connecting two highly-connected nodes is more crucial for information flow in the network than simply a high-degree node. These results clearly suggest the importance of exploiting the topology of the network for controlling the dynamics. It may be noted that for ten realizations of the scale-free network of size $N = 1000$, we observed that the pinning strength required for achieving control for a particular pinning density differed by $\pm 0.04$ in each case.

In Fig. 3.10 is depicted the weakly chaotic spatiotemporal dynamics on pinning high-centrality nodes in a scale-free network.
Analysis of underlying topology on dynamical networks

Fig. 3.10: Spatiotemporal dynamics on a scale-free network in the low chaos ($r = 3.6$) regime ($\lambda_{\text{max}} = 0.18$). (a) Top 15% high-degree nodes pinned with $p_s = -0.30$ ($\lambda_{\text{max}} = -0.07$), (b) 5% high-betweenness nodes pinned with $p_s = -0.22$ ($\lambda_{\text{max}} = -0.139$), (c) 5% high-closeness nodes pinned with $p_s = -0.22$ ($\lambda_{\text{max}} = -0.138$), (d) 10% nodes having high values of eigenvector for Laplacian matrix, with $p_s = -0.25$ ($\lambda_{\text{max}} = -0.09$).

We next assess the control of strongly chaotic spatiotemporal dynamics by pinning high-centrality nodes in scale-free network. It is clear from Fig. 3.11 that in this case higher pinning strengths ($> -0.3$) at much higher pinning densities (~25%) are required to achieve global control with high-betweenness or high-closeness nodes (shown in Fig 3.11(b) and 3.11(c) respectively) as compared to the low chaos regime. As is clear from Fig. 3.11(a), no control of spatiotemporal dynamics is observed on pinning even 50% of high degree nodes while is achievable by pinning top 25% of the nodes having high eigenvector of the Laplacian matrix (Fig. 3.11(d)). Thus, choice of nodes for applying external perturbation has a significant effect in controlling the dynamics on networks.
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![Graphs showing spatiotemporal dynamics](image)

**Fig. 3.11:** Spatiotemporal dynamics on a scale-free network in the high chaos ($r = 3.9$) regime ($\lambda_{\text{max}} = 0.42$). (a) Top 50% high-degree nodes pinned with $p_s = -0.40$ ($\lambda_{\text{max}} = 0.02$), (b) 25% high-betweenness nodes pinned with $p_s = -0.32$ ($\lambda_{\text{max}} = -0.019$), (c) 25% high-closeness nodes pinned with $p_s = -0.38$ ($\lambda_{\text{max}} = -0.05$), (d) 50% nodes having high values of eigenvector for Laplacian matrix pinned with $p_s = -0.35$ ($\lambda_{\text{max}} = -0.04$).

### 3.1.2.2.3 Random network:

In this section, the results for control by pinning on random networks are presented. These results have been compared to pinning randomly chosen nodes in regular, small-world and scale-free networks. Due to the absence of regularity in connectivity of the nodes, the scenario where regularly spaced nodes are pinned will not be considered in this case. Unlike small-world network which has few long-range connections and scale-free network which has a few nodes having very high connectivity, random networks have completely random connectivity of nodes. Hence, centrality measures do not play any role in this case.

When the dynamics of the network is weakly chaotic, at least 20% of the nodes chosen randomly are required for pinning with a pinning strength of -0.30 to achieve suppression of STC. While no control could be achieved by pinning 20% randomly chosen nodes in regular and small-world networks, a higher pinning strength is required in case of scale-free networks. This could be because some of the nodes chosen randomly in scale-free network may also be nodes having very high values of centrality measures. When the dynamics is highly chaotic, control of STC can be achieved by pinning at least 50% of the nodes chosen randomly with a pinning strength of -0.27. No control is observed by pinning nodes randomly in regular, small-world and scale-free networks when the dynamics is highly chaotic. For different configurations of the random network, it was found that the pinning density required for achieving control is the same but the pinning strength varied by ±0.04.
3.1.2.3 Pinning randomly chosen nodes: In many situations one may have no information about the underlying connectivity topology in real dynamical systems. In such situations, the most obvious choice would be to choose nodes randomly for exerting pinning. Also this presents us a control for the various pinning strategies discussed above based on the geometrical properties of the network topology. Hence we now consider the efficacy of our approach to suppress chaos on the four topological networks for a random distribution of pinning nodes. In this case, we consider pinning nodes randomly at various pinning densities and analyze the minimum strength of pinning required to suppress chaos. The results of this analysis are summarized in Table 3.6 (averaged over 10 realizations).

Table 3.6: Comparison of pinning strengths on randomly chosen sites on four topological networks to achieve global control.

<table>
<thead>
<tr>
<th>r</th>
<th>pd</th>
<th>50%</th>
<th>20%</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.6</td>
<td>Reg</td>
<td>-0.05</td>
<td>NC</td>
</tr>
<tr>
<td></td>
<td>SW</td>
<td>-0.05</td>
<td>NC</td>
</tr>
<tr>
<td></td>
<td>SF</td>
<td>-0.04</td>
<td>-0.21</td>
</tr>
<tr>
<td></td>
<td>Rand</td>
<td>-0.037</td>
<td>-0.3</td>
</tr>
<tr>
<td>3.9</td>
<td>Reg</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td></td>
<td>SW</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td></td>
<td>SF</td>
<td>NC</td>
<td>NC</td>
</tr>
<tr>
<td></td>
<td>Rand</td>
<td>-0.27</td>
<td>NC</td>
</tr>
</tbody>
</table>

In the event of the system exhibiting weak chaos, control by random pinning is achievable in all the four networks but at a very high pinning density ($p_d = 50\%$). In the case of scale-free and random networks control is achievable at much lower pinning densities ($p_d = 20\%$). In the strong-chaos regime, suppression of chaos is observed only in the case of random networks with 50\% of the nodes pinned. It may be noted that in the case of small-world topology, pinning randomly chosen nodes controls the global dynamics at much lower pinning strength ($p_s = -0.05$) compared to pinning high-degree nodes (shown in Figs. 3.9(c) and (d)). This may be because since very few nodes have high degree, most of the nodes are randomly chosen, and so the results may be considered as two different random configurations. In the case of scale-free networks, pinning 15\% high-degree nodes suffices in achieving control with a pinning strength of -0.30 compared to 20\% randomly chosen nodes required to achieve control with a pinning strength of -0.21. This can be attributed to some of the randomly chosen nodes also having high centrality values. On choosing a different random set of 20\% nodes, a higher or lower pinning strength may be required depending upon whether the nodes chosen also have
high centrality values. In case of degree, control is certain for a given pinning strength and density, while for randomly chosen nodes, it will depend on the chosen random nodes.

Hence, we summarize that in the case of small-world networks, suppression of STC is attainable at a minimum density of 20% of external probes when the dynamics is weakly chaotic and the probes are uniformly arranged at regular intervals. With the increase in complexity of the dynamics, a finer grid of external probes is required. Choosing nodes based on high values of centrality measures does not offer any advantage in this case. In case of scale-free network, control of STC is attainable at very low pinning density (~5%) on choosing high betweenness or high closeness nodes, when the dynamics is weakly chaotic. As expected higher density and strength of pinning is required with increasing complexity of dynamics. Thus, we conclude that pinning regularly spaced nodes in small-world network and pinning high betweenness and closeness nodes in scale-free networks requires the minimum number of nodes to be pinned.

3.2 Neuron Map

In the previous chapter, we have shown the wide variety of dynamical behaviour exhibited by the neuron map (Eqn. 2.14): from stable to chaotic with intermittent periodic behaviour as a function of $k$. We also discussed construction of a neuronal network and by defining the dynamics on each of its nodes by a neuron map to mimic the excitations in brain and its connectivity pattern. The brain is known to have structural and functional features of complex networks – such as small-world topology, highly connected hubs and modularity. Since a neuronal network is known to have strictly non-local coupling between the neurons (Bullmore and Sporns, 2009), here we analyze neuronal network modeled on a small-world and scale-free network topology. Hence, in the following sections, we first discuss the dynamics of the coupled neuronal networks for the two topologies. Then discuss how we can manipulate or control the dynamics on a single neuron map followed by our results for a coupled dynamical system.

**Dynamics of Coupled Maps:**

We first analyze the spatiotemporal behaviour of neuron maps coupled on a small-world and a scale-free topology. The bifurcation diagram of the variable $x$, the neuron membrane potential, with respect to $k$, (a constant bias) is plotted for a representative node for 200 time steps after removing the transients on the small-world and scale-free network topologies (shown in Fig 3.12). Figure 3.13 exhibits the spatial behaviour for the two networks plotted against $k$. 

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Fig. 3.12: Bifurcation diagram w.r.t. $k$ for a representative node in a neuronal network with $\varepsilon = 0.05$ (a) small-world and (b) scale-free topology. 200 time steps have been plotted after removing the transients.

Fig. 3.13: Spatial behaviour w.r.t. $k$ for all the nodes in a neuronal network with $\varepsilon = 0.05$ (a) small-world and (b) scale-free topology plotted for the last time step.

From Fig. 3.12, we observe that the asymptotic temporal behaviour of the two networks is very similar. But spatial behaviour being governed by the local dynamical function shown in Fig. 3.13, we observe that the range of dynamics in both the cases is different. Scale-free network exhibits a smaller range of $x$-values as a function of $k$ differs in the two cases. The asymptotic temporal behaviour on both networks differs significantly from the single neuron map dynamics (Fig 2.12) as a result of coupling. Hence, topology plays a role in the dynamics of the network.
In an earlier study on synchronization of network dynamics with increasing coupling strength (Jampa et al., 2007), it was shown that increasing randomness in the coupling connections between nodes in a network aids synchronization. In Figure 3.14, we present the bifurcation diagram w.r.t. coupling strength $\varepsilon$, for the two topological networks in the chaotic ($k = 0.03$) regime.

![Bifurcation Diagram](image)

**Fig. 3.14:** Spatial behaviour of coupled neuron maps as a function of coupling strength, $\varepsilon$, in the high chaos regime ($k = 0.03$) for (a) Small-world, and (d) Scale-free network. 200 time steps have been plotted after removing 1000 transients.

It is clear from Fig. 3.14(a) that the small world network attains a synchronized state for $\varepsilon > 0.36$. The scale-free network attains spatial synchronization for $\varepsilon > 0.1$ (Fig 3.4(b)). Hence, for generating spatiotemporally chaotic dynamics, we have chosen a low value of coupling strength, $\varepsilon = 0.05$.

In normal functioning of the brain, the neuronal dynamics is chaotic. During perception and under pathological conditions such as epileptic seizures and hallucinations, the chaotic dynamics of the brain becomes periodic in nature (Shuai et al., 1996). Hence, it is of practical significance to be able to manipulate the dynamics of a neuronal network in such a way that its periodic dynamics can be taken to a chaotic state. In studying the control of coupled map dynamics, by fragmenting the system into smaller units, we can first analyze the dynamics on a simpler unit and then map it to study a more complex network. Hence, we first aim to study the control or manipulation of dynamics of a single neuron map.

**Controlling on Single Neuron Map Dynamics:**

The effect of pinning on a single neuron map, we showed that in the absence of external perturbation ($k = 0$), it exhibits a fixed point dynamics (Fig. 2.12(a)) (it is in the resting state). As
shown in Fig 3.15(a), on applying a positive constant pinning signal, $p_s$, the system, initially at $k = 0$ can be taken from a stable to any chaotic state, similar to the bifurcation behaviour w.r.t. $k$ in the absence of pinning (Fig 3.15(a)). That is, for a single neuron map, $p_s$ basically modulates the $k$ parameter. Similarly, when the system is at $k = 0.08$, negative pinning can generate the same bifurcation behaviour w.r.t. $k$ (Fig 3.15(b)).

Thus, we can manipulate the neuron dynamics by varying the strength and sign of the pinning signal depending on the initial and desired final dynamical state of the neuron. In our study of control of dynamics of the neuron map, we will be considering the neuronal dynamics in two parameter regimes (1) chaotic ($k = 0.03$) and (2) periodic ($k = 0.048$). This is in accordance with real practical situations, where one is interested in taking the highly periodic dynamics of brain (under pathological conditions such as epileptic seizures, hallucinations), to a state of normal functioning, $i.e.$, chaos.
From Eqn. 2.14, it is clear that if a pinning signal, $p_s$ is applied to the $x$-variable, the magnitude and the sign of this signal can be chosen based on the value of $k$ and the target state of dynamics we wish to achieve. In other words, since the pinning term, $p_s$ is just an additive constant to parameter $k$, the system dynamics can be manipulated by appropriately choosing $p_s$ for a given value of $k$. For instance, if we intend to take the dynamics at $k = 0.048$ to a target state of dynamics corresponding to $k = 0.03$, then $p_s = -0.018$ will be required.

**Control on Coupled Map:**

Brain is a highly complex network of neurons interacting with non-local connectivity. We now discuss our results on controlling the dynamics by applying pinning to all or few nodes in a neuronal network with small-world and scale-free topology. Here we shall discuss anti-control, i.e., to take the system from a periodic (corresponding to $k = 0.048$) to a chaotic state ($k = 0.03$). Like in the case of logistic map, we discuss uniform and non-uniform pinning and assess the controllability.

**3.2.1 Uniform Pinning**

All the nodes in the small-world and scale-free networks are pinned with the same pinning strength in this case. This trivial study gives an idea about the strength of pinning required to control the dynamics on the network. The spatiotemporal behavior of variable $x$ at $k = 0.048$ for a small world network when no pinning is applied ($\lambda_{\text{max}} = -0.134$) is shown in Fig. 3.16(a) and in Fig 3.16(b) when pinning is applied to all the nodes in the network with $p_s = -0.018$ ($\lambda_{\text{max}} = 0.119$). The value of $\lambda_{\text{max}}$ for the spatiotemporal behaviour of small-world network in chaotic regime ($k = 0.03$) is 0.125, which is quite close to the $\lambda_{\text{max}}$ value when pinning is applied to all nodes in the network with a pinning strength of -0.018. Thus, we can say that the periodic dynamics of the network can be taken to a chaotic state by pinning all nodes uniformly.
Analysis of underlying topology on dynamical networks

Fig. 3.16: Spatiotemporal behaviour when (a) no pinning ($\lambda_{\text{max}} = -0.134$), (b) pinning with strength, $p_s = -0.018$ ($\lambda_{\text{max}} = 0.119$), is applied to all nodes for $k = 0.048$ and $\varepsilon = 0.05$, in a small-world network. This has been plotted for all nodes in the network for 200 time steps after removing the transients.

Now, we show the effect of pinning all the nodes in a scale-free network. The spatiotemporal behavior at $k = 0.048$ for a scale-free network when no pinning is applied ($\lambda_{\text{max}} = -0.145$) is shown in Fig. 3.17(a) and in Fig 3.17(b) when pinning is applied to all the nodes in the network with a strength $p_s = -0.018$ ($\lambda_{\text{max}} = 0.043$). The value of $\lambda_{\text{max}}$ for the spatiotemporal behaviour of scale-free network in chaotic regime ($k = 0.03$) is 0.05, which is close to the $\lambda_{\text{max}}$ value when pinning is applied to all nodes in the network with a pinning strength of -0.018. Thus, we can say that the periodic dynamics of the network can be taken to a chaotic state by pinning all nodes uniformly.
Analysis of underlying topology on dynamical networks

Fig. 3.17: Spatiotemporal behaviour when (a) no pinning \( \lambda_{\text{max}} = -0.145 \), (b) pinning with strength, \( p_s = -0.018 \) \( \lambda_{\text{max}} = 0.043 \), is applied to all nodes for \( k = 0.048 \) and \( \varepsilon = 0.05 \), in a scale-free network. 200 time steps are plotted after removing the transients.

From Fig. 3.16(a) and Fig. 3.17(a), it is clear that the dynamics of the small-world and scale-free network respectively at \( k = 0.048 \), is periodic. From Fig 3.16(b) and Fig. 3.17(b), spatiotemporally chaotic dynamics is shown as indicated by \( \lambda_{\text{max}} \). We can see that in small-world and scale-free networks respectively, on pinning all the nodes in the two networks with \( p_s \sim -0.018 \), chaos can be induced.

3.2.2 Non-Uniform Pinning

Since it is not feasible to apply electric pulses on a very fine grid over the entire brain, we next consider the possibility of controlling the neuronal excitations by pinning a few selected nodes in the neuronal network. The following pinning strategies have been considered.

1. Regularly spaced nodes: In case of small-world network, since there is an inherent ordering in the arrangement of its nodes, we apply pinning at regularly spaced intervals.

2. Randomly chosen nodes: Nodes are chosen randomly for pining in both the networks.

3. High Centrality measure nodes: As discussed in Section 3.1.2.2, various centrality measures capture different geometrical/topological properties of the networks with non-regular topology. They help us in identifying nodes which are critical to the existence of the network. Hence, we consider the effect of pinning high centrality nodes in the case of two networks. First we present
our results for pinning nodes non-uniformly in the case of small-world network and then for the scale-free network.

### 3.2.2.1 Small-World Network:
We present our analysis of controlling neuronal dynamics on a small-world network and compare the pinning densities and pinning strengths required to induce chaos when nodes in the network are regularly spaced and when they are randomly chosen. Then we consider pinning of nodes having high values for the centrality measures degree, betweenness, closeness, and eigenvector for adjacency and Laplacian matrices. In Table 3.7, we have summarized the results of our analysis.

**Table 3.7:** Comparison of pinning densities and pinning strengths required to induce global chaos when the small-world network is exhibiting periodic dynamics \((k = 0.048)\)

<table>
<thead>
<tr>
<th></th>
<th>(p_d)</th>
<th>(p_s)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Regularly Spaced Nodes</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>-0.02</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>-0.025</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>NC</td>
<td></td>
</tr>
<tr>
<td><strong>Randomly Chosen Nodes</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>50%</td>
<td>-0.023</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>-0.027</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>NC</td>
<td></td>
</tr>
<tr>
<td>(k)</td>
<td>30%</td>
<td>-0.032</td>
</tr>
<tr>
<td>(b)</td>
<td>30%</td>
<td>-0.032</td>
</tr>
<tr>
<td><strong>CI</strong></td>
<td>50%</td>
<td>NC</td>
</tr>
<tr>
<td><strong>(A_{lev})</strong></td>
<td>50%</td>
<td>NC</td>
</tr>
<tr>
<td><strong>(L_{lev})</strong></td>
<td>50%</td>
<td>NC</td>
</tr>
</tbody>
</table>

From Table 3.7, it is clear that a minimum pinning density of 20% suffices in inducing chaos when the nodes are chosen for pinning at spaced at regular intervals or are randomly distributed. The strength of pinning may vary depending on the initial state of the system and the desired target state. However, in the case of random distribution of pinning sites, the desired dynamical state may not always be guaranteed.

On choosing nodes based on centrality measures, we observe that higher density of pinning \(\sim 30\%\) on pinning high degree or high betweenness nodes, in order to induce chaos. Pinning the other three centrality measures was not able to induce chaos even at 50% pinning density. In Fig. 3.18, we show the spatiotemporal behaviour of the small-world network when pinning is applied at varying pinning densities and strengths on regularly spaced nodes, randomly chosen nodes and nodes having high values of centrality measures.
Analysis of underlying topology on dynamical networks

Fig. 3.18: Spatiotemporal behaviour of the small-world network with $k = 0.048$ and $\varepsilon = 0.05$ on pinning (a) every fifth node with $p_s = -0.!025 (\lambda_{\text{max}} = 0.09)$, (b) 20% randomly chosen nodes with $p_s = -0.027 (\lambda_{\text{max}} = 0.11)$, (c) 30% high degree nodes with $p_s = -0.032 (\lambda_{\text{max}} = 0.105)$, (d) 30% high betweenness nodes with $p_s = -0.032 (\lambda_{\text{max}} = 0.128)$. 200 time steps have been superimposed after eliminating the transients.

As in the case of suppressing chaos in logistic network dynamics, we observe that inducing/enhancing chaos is easily attainable on pinning regularly spaced nodes in a small-world network compared to randomly chosen nodes or nodes chosen based on their centrality values.

3.2.2.2. Scale-Free Network

In a scale-free network, there is an absence of local and nearest-neighbour connectivity seen in small-world network. Hence, in this case we do not consider pinning of regularly spaced nodes. We present our results for pinning density and pinning strength required for inducing chaos when the system is in periodic state ($k = 0.048$), by pinning randomly chosen nodes and nodes chosen based on high values of centrality measures. There results have been summarized in Table 3.8.

Table 3.8: Comparison of pinning densities and pinning strengths required to induce global chaos by pinning nodes chosen at random or based on high centrality values in a Scale-free network ($k = 0.048$)

<table>
<thead>
<tr>
<th>Randomly Chosen Nodes</th>
<th>$p_d$</th>
<th>$p_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50%</td>
<td>-0.023</td>
<td></td>
</tr>
<tr>
<td>20%</td>
<td>-0.028</td>
<td></td>
</tr>
<tr>
<td>10%</td>
<td>NC</td>
<td></td>
</tr>
<tr>
<td>k</td>
<td>15%</td>
<td>-0.006</td>
</tr>
<tr>
<td>b</td>
<td>12.5%</td>
<td>-0.006</td>
</tr>
<tr>
<td>Cl</td>
<td>12.5%</td>
<td>-0.006</td>
</tr>
<tr>
<td>A_{lev}</td>
<td>15%</td>
<td>-0.006</td>
</tr>
<tr>
<td>L_{lev}</td>
<td>12.5%</td>
<td>-0.005</td>
</tr>
</tbody>
</table>
From Table 3.8, it is clear that choosing nodes based on high centrality measures offers a great advantage in terms of both pinning density and pinning strength. By pinning as few as 12.5% of the high betweenness, high closeness nodes or nodes having high value of eigenvectors of Laplacian matrix, we are able to induce chaos and take the dynamics from a highly periodic state \( (k = 0.048) \) to a chaotic state. However, on choosing nodes at random for pinning, a much higher pinning strength and density is required. Also, the desired dynamical state may not be achieved. Thus, we clearly see that by choosing a small number of nodes based on their high centrality values, clearly provides an advantage over randomly chosen nodes to induce/enhance chaos in the dynamics.

Figure 3.19 shows the spatiotemporal behaviour of a scale-free network in the periodic regime \( (k = 0.048) \) on pinning various high-centrality nodes. It is clear from the figure that a much lower pinning strength is required to induce chaotic dynamics in a scale-free network compared to small-world network. The pinning strength and pinning density required is similar for all the centrality measures analyzed, degree (Fig 3.19(a)), betweenness (Fig 3.19(b)), closeness (Fig 3.19(c)) and eigenvector of Laplacian matrix (Fig 3.19(d)).

![Fig. 3.19](image)

**Fig. 3.19:** Spatiotemporal behaviour of the scale-free network shown for \( k = 0.048 \) and \( \epsilon = 0.05 \) for (a) 15% high degree nodes pinned with \( p_s = -0.006 \) \( (\lambda_{\text{max}} = 0.041) \), (b) 12.5% high betweenness nodes pinned with \( p_s = -0.006 \) \( (\lambda_{\text{max}} = 0.039) \), (c) 12.5% high closeness nodes pinned with \( p_s = -0.006 \) \( (\lambda_{\text{max}} = 0.039) \), (d) 12.5% nodes having high value of eigenvector for Laplacian matrix pinned with \( p_s = -0.005 \) \( (\lambda_{\text{max}} = 0.038) \). 200 time steps plotted after eliminating the transients.

Hence, we can summarize that in an attempt to induce chaos to the highly periodic dynamics in small-world topology of the neuronal network, choosing regularly spaced nodes
works the best as compared to choosing nodes randomly and choosing nodes based on high centrality measures. This can be attributed to an inherent ordering in the connectivity pattern of the nodes in the small-world network. Primarily, the coupling being local in nature, with a very few long-range interactions; the effect of pinning regularly spaced nodes can be seen to induce chaos in highly-periodic global dynamics across the network. In case of scale-free network, pinning only 12.5% nodes having high values of degree, betweenness and eigenvector corresponding to the Laplacian matrix helps in inducing chaos in the global dynamics. On the other hand, at least 20% randomly chosen nodes are required for pinning with a much higher pinning strength. Hence, making use of centrality measures to identify nodes for pinning works well in case of scale-free networks while pinning regularly spaced nodes works in case of small-world networks.

Thus, we again show that exploiting the geometrical topology helps in inducing chaos in the system exhibiting periodic dynamics due to the dominance of strong nearest-neighbour interactions in small-world network, pinning regularly spaced nodes takes the system to any desired dynamical state for very low pinning densities. In the case of scale-free networks with non-homogeneous connections, pinning nodes “critical” for the stability and integrity of the network, which are defined by centrality measures of the connection topology are most advantageous in attaining the desired state. The pinning strengths for control are about ten times lower for scale-free network compared to small-world network. This may be due to the high efficiency of the scale-free network compared to small-world network. This has important application in neuronal networks, the brain cortical topology being scale-free; regulating the neuronal excitations can be easily achieved by very small external perturbations.
4 Conclusion

Many physical and biological systems exhibit a variety of spatiotemporal dynamics—from stable to chaotic—that can change under pathological conditions and impair their normal functioning. With the advancements in computing power, it has been found that the connectivity topology in these systems need not always be nearest-neighbor diffusive coupling. They may exhibit heterogeneity in the spatial coupling, resulting in nodes having long-range connectivity, e.g., in airport network and/or being densely coupled, e.g., in World Wide Web. These systems can be modeled as networks/graphs. In most earlier studies, spatially extended dynamical systems have been modeled as a coupled map lattice where each node is a dynamical entity and is diffusively coupled to its nearest neighbours in a fixed, ordered connectivity pattern. However, a large number of physical and biological systems, e.g., excitable tissues, are shown to exhibit non-regular, non-local connection pattern. Thus, being able to manipulate the altered dynamics for improved functioning on different connection topologies has potential for wide ranging applications in real and artificial systems. With this objective, in this study, we have assessed the efficacy of global controllability by the pinning approach in spatially extended systems whose local dynamics is regulated by coupled multi-variable processes, coupled to its neighbours in four different connection topologies, viz., regular, small-world, scale-free and random. This method is easy to implement and does not require any a priori information of the system dynamics or explicit changes in its parameters.

The control and anti-control of chaotic dynamics has important implications in biological systems. For example, in the human heart, an excitable group of cells called the ectopic pacemaker or the ectopic focus causes a premature heart beat outside the normally functioning sinoatrial (SA) node. In a healthy heart, the SA node usually suppresses the activity of the ectopic pacemaker owing to its higher impulse rate. However, in the instance of either a malfunctioning SA node or the ectopic foci bearing an intrinsic rate superior to SA node rate, ectopic pacemaker activity may rule over the heart rhythm causing pathological conditions. Thus, in this case, it is desirable to suppress the chaotic dynamics of the ectopic focus. In certain situations, for example in the brain tissue, it may be desirable to enhance or induce chaotic dynamics under pathological conditions, viz., epileptic seizures. In neuronal networks, it is known that the resting state of dynamics is chaotic. During arousal/attention, perception, or
pathological conditions such as epileptic seizures and hallucinations, the chaotic dynamics becomes to periodic. Hence it is of practical significance to manipulate the dynamics in the neuronal network such that the dynamics can be brought back from a state of periodicity to a state of chaos (Anti-control). The topology of the brain has been shown to be neither completely regular nor completely random in nature. Hence, to simulate the neuronal dynamics, in the thesis, we have defined the dynamics on each node to be governed by Chialvo’s neuron map, which models an excitable system, and the connection topology considered is small-world and scale-free.

In this thesis, we have considered two mathematical models for defining the local dynamics on each node of the networks considered. The first is the logistic map which is a single dimensional map, exhibits a wide variety of dynamics including chaos. The other is the neuron map which is a two-dimensional map, which models an excitable system. The control approach here involves applying a constant perturbation or pinning signal to the nodes in the network. Here we propose to exploit the connection topology for identifying nodes for applying external perturbation. For a regular network (coupled map lattice model), the local nearest-neighbour connection topology dictates choosing nodes at a fixed interval for achieving control. In the case of networks with non-local and heterogenous coupling topology, we have used concepts from graph theory to provide a rationale for identifying critical nodes for pinning. Thus, depending upon the topology of the network, different pinning strategies have been discussed and the efficacy of each of the four topological networks analyzed. Following situations have been analyzed – uniformly pinning all nodes in the network, pinning regularly spaced nodes (e.g. in the case of regular and small-world networks), pinning nodes identified based on centrality measures (e.g. small-world and scale-free networks), and pinning randomly chosen nodes (useful when no information of the connection topology is available. The various centrality measures considered in this study are degree, betweenness, closeness, and eigen spectra of adjacency and Laplacian matrices.

When the local dynamics is governed by logistic map, we have presented our analysis for the system exhibiting weak chaos \((r = 3.6)\) and strong chaos\((r = 3.9)\). We find that in both the parameter regimes, by uniformly pinning all the nodes in the network, spatiotemporally chaotic dynamics can be controlled to any desired dynamical state irrespective of the underlying topology. However, the pinning strength required for control is higher for \(r = 3.9\) compared to \(r\)
= 3.6, as expected. But pinning all the nodes in the network is not practically feasible. Hence, we next analyzed the control of dynamics when the nodes are pinned at regular intervals and the network topology is regular and small-world. We observe that in the low chaos regime, pinning 20% of the nodes suffices, while 50% of the nodes are required to be pinned in the high chaos regime for controlling the dynamics in both regular and small-world networks. In the case of small-world network topology, we also considered pinning nodes based on their centrality values. Control was achieved by pinning only high degree and high betweenness nodes, however, the pinning strength required was very high compared to pinning regularly spaced nodes. This is because of the inherent ordering in the connectivity pattern of the nodes in the small-world network. Almost all the nodes are locally coupled with very few nodes having long-range interactions. As a result, the high degree nodes do not really have very large number of connections compared to the average degree of 6. Most high degree nodes are randomly chosen as they all have the same degree and so the results are actually comparable to randomly chosen nodes. Thus, by exploiting the geometrical topology, which in the case of small-world networks, is a well-ordered, nearest-neighbour connections, pinning at regular intervals is most efficient in achieving control. In scale-free networks, to exploit the connection topology, we only consider pinning nodes identified based on various graph centrality measures. Since there is no inherent order in the connectivity amongst nodes, pinning at regular intervals is not applicable here. Pinning as few as 5% of the high betweenness and high closeness nodes in the low chaos regime and 25% of the high betweenness and high closeness nodes in the high chaos regime suffices in achieving control of dynamics in this case. Thus, in the case of scale-free networks, centrality measures are very useful in identifying nodes for pinning. This is in accordance with number of studies on the effect of removing high centrality nodes on the stability of the network. Targetted removal of nodes can collapse the network while random removal of nodes have little or no impact on the integrity of the network. Similarly, pinning randomly chosen nodes requires much higher pinning strengths compared to pinning high-centrality nodes for obtaining control of the network dynamics, and is dependent on a chosen random configuration. Random networks, which do not have any characteristic connectivity pattern, is considered here as a control.

It has been shown that most real-world systems operate in the low chaotic regime while fully developed chaos is not very common. We showed that very low pinning strengths and pinning densities, are required for suppressing chaos, on both small-world and scale-free
networks, when the local chaos is weakly chaotic. Thus, the approach proposed here of exploiting geometrical topology for applying external perturbation is of practical significance and can be easily applicable to real-world systems.

In our analysis of neuronal network dynamics, we have presented results only for inducing/enhancing chaos, i.e., anti-control. In this case also, low pinning density (i.e., every fifth node) suffices in inducing chaos in both small-world and scale-free networks. Also, on pinning high centrality nodes, much lower pinning density and pinning strength is required for manipulating the dynamics on scale-free network compared to small-world network.

It is clear from the study that the network topology does play a significant role in governing the dynamics of spatially extended systems and should be considered while applying external perturbation. Thus, analysis of the connection topology in any dynamical system can help in choosing a minimal set of nodes for pinning, considerably reducing the effort required for manipulating the system dynamics.
Bibliography


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