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by

Avni Verma, Kamalakar Karlapalem

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Design and Evaluation of Alternate Enumeration Techniques for Subset Sum Problem

AVNI VERMA and KAMALAKAR KARLAPALEM, International Institute of Information Technology, Hyderabad, India

The subset sum problem, also referred as SSP, is a NP-Hard computational problem. SSP has its applications in broad domains like cryptography, number theory, operation research and complexity theory. An algorithm for solving SSP is Backtracking Algorithm which has exponential time complexity. Our goal is to design and develop better alternate enumeration techniques for faster generation of SSP solutions. Given the set of first n natural numbers which is denoted by X_n and a target sum S, we design alternate enumeration techniques which find all the subsets of X_n that add up to sum S.

In this paper, we (i) analyze the distribution of power set of X_n and derive formulas which show patterns and relations among these subset. (ii) introduce three major distributions for power set of X_n: Sum Distribution, Length-Sum Distribution and Element Distribution. These distributions are prepossessing procedures for alternate enumeration techniques for solving SSP. (iii) propose novel algorithms: Subset Generation using Sum Distribution, Subset Generation using Length-Sum Distribution, Basic Bucket Algorithm, Maximum and Minimum Frequency Driven Bucket Algorithms and Local Search using Maximal and Minimal Subsets for enumerating SSP.

We compare the performance of these algorithms against the traditional backtracking algorithm. The efficiency and effectiveness of these algorithms are presented with the help of these experimental results. Furthermore, we studied number of subsets generated by algorithms over the solution size to evaluate efficiency of algorithms. Finally, we present a conjecture about upper bound on the number of subsets that has to be enumerated to get all solutions for Subset Sum Problem.

Additional Key Words and Phrases: Subset Sum Problem, Algorithms

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1 INTRODUCTION

In SSP, we consider a set of n positive integers stored in set X and a target sum S. $X = \{x_1, x_2 \ldots x_n\}$. Traditionally, there are two definitions for SSP which are described below:

(1) Version 1: Given a set X containing positive integers and a target sum S, is there a subset which sum upto S? This is a NP-Complete problem.

For example, given $X = \{5, 4, 9, 11\}$ and $S = 9$, the solution to this problem is *true*. There are many ways to solve this problem and it depends on the size and values of X and S. The brute force algorithm iterates through all possibilities and takes $O(2^n \times n)$ time for execution. For smaller size and values of X and S, SSP can be solved in polynomial time by using dynamic programming with time complexity $O(n \times S)$ [19].

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(2) Version 2: Given a set X containing positive integers and a target sum S, find a subset which can sum up to S. This is a NP-Hard problem.

For $X = \{5, 4, 9, 11\}$ and $S = 9$, the solution to above problem is either $\{5, 4\}$ or $\{9\}$. This is an exponential time taking problem which can be solved in $O(2^n \times n)$ time by using brute force method. This method requires $O(n)$ storage space to store the required result. This version of SSP does not have any known polynomial time algorithm.

In this paper, we extend the traditional SSP (Version 2) and design alternate enumeration techniques. Instead of finding one subset with target sum, we find all possible solutions of SSP. Therefore, for $X = \{5, 4, 9, 11\}$ and $S = 9$, solution to our version of SSP are $\{5, 4\}$ and $\{9\}$. We further confine and refine our problem domain by considering first n natural numbers as set X. There are many advantages for selecting this problem domain. It avoids duplications and hence simplifies the problem statement. Since the sum of first n natural number is $\frac{n(n+1)}{2}$, by selecting $X = \{1, 2 \ldots n\}$ we restrict target sum between 1 and $\frac{n(n+1)}{2}$, $S \in [1, \frac{n(n+1)}{2}]$. The efforts to solve Subset Sum Problem are required to get subset queries in relational databases [18]. Before describing the formulation of our problem in detail we present the research in the field of SSP.

2 RELATED WORK

The Subset Sum Problem has been studied very widely. It has a standard $O(nu)$ pseudo-polynomial time dynamic programming algorithm [17] which is taught in elementary algorithms class. Here, n is the number of elements of the set thats sums to u. Additionally, there are a number of other algorithms in the literature, including an FPTAS [10], an exact algorithm with space and time trade offs [1], a polynomial time algorithm for most low density sums [14], and a number of more specialized pseudo-polynomial time algorithms with various properties [12] [5] [15] [13].

There is also another variant of Subset Sum Problem which allows the elements in X to be used any number of times in the sums. Overall, dynamic programming is expected to be most efficient for very dense instances, while backtracking is expected to be most efficient for sparse instances of Subset Sum Problem.

In [11], the authors have introduced a new faster pseudo-polynomial time algorithm for the Subset Sum problem to decide if there exists a subset of a given set S whose elements sum to a target number t. Their proposed algorithm runs in $\tilde{O}(\sqrt{n}t)$ time, where n is the size of set S. This paper further generalizes the problem by computing this for all target numbers $t \leq u$ in $\tilde{O}(\sqrt{n}u)$ time.

Despite the apparent simplicity of the problem statement, to date there has been modest progress on exact algorithms[4] for Subset Sum Problem. Indeed, from a worst-case performance perspective the fastest known algorithm runs in $O(2^{\frac{n}{2}})$ time and dates to the 1974 work of Horowitz and Sahni[9]. Improving the worst-case running time is a well-established open problem[20] [2].

In [2], the authors present a randomized algorithm. They consider positive integers and a target sum but instead of finding all subsets of target sum, the solution is bounded by B concentration. The main result of this algorithm is that all instances without strong additive structure (without exponential concentration of sums) can be solved faster than the Horowitz-Sahni time bound $O(2^{\frac{n}{2}})$ [9]. They have also shown a quantitative claim to show or prove it. Complexity of this randomized algorithms is $O(2^{0.3399n}B^4)$.

Beier and Vocking [3] presented an expected polynomial time algorithm for solving random knapsack instances. Knapsack and subset sum have similarities, but the random instances considered there are quite different from ours, and this leads to the development of quite a different approach. Subset sum problem is also closely related to the classical number theory study of determining partitions. In [8] Hardy and Wright provide generating functions but is limited due to lack of computational scheme for generating such partitions [9]. A survey of algorithms for the different variations of the knapsack problem is given in [9]. Much of the early work in the knapsack problem was done by Gilmore and Gomory [7] [6].

However, there is very little work done on enumeration techniques for subset sum problem, which we addressed in this work. We have developed different algorithms for alternate enumerations techniques for subset sum problem and have compared their performance.

3 FORMULATION FOR SUBSET SUM PROBLEM

The following set of information is used for presenting the exponential aspect and solution of alternate enumeration techniques of SSP:

(1) A set of first n natural numbers. $X_n = \{1, 2, 3 \ldots n\}$ where n is a positive integer. The set X_n is also known as the *Universal set*. The cardinality of the set X_n is n. $|X_n| = n$

(2) A set of all subsets of X_n is $\mathcal{P}(X_n) = \{\phi, \{1\}, \{2\} \ldots \{1, 2 \ldots n\}\}$. It is also known as power set. The empty set is denoted as ϕ or {} or the null set. In this paper, we use ϕ for the representing empty set. $|\mathcal{P}(X_n)| = a = 2^n$

(3) $maxSum(n)$ is the sum of all elements of the universal set X_n. This is the maximum possible sum for any element of $\mathcal{P}(X_n)$.

$maxSum(n) = b = (1 + 2 + 3 \ldots n) = \frac{n(n+1)}{2}$.

$Sum(A) \leq maxSum(n) = \frac{n(n+1)}{2}\ \forall A \in \mathcal{P}(X_n)$

(4) $Sum(A)$ is the sum of all elements of a set A where A belongs to power sets of X_n, $A \in \mathcal{P}(X_n)$.

- We assume sum of all elements of ϕ as 0, $Sum(\phi) = 0$.
- The range of $Sum(A)$ is $[0, \frac{n(n+1)}{2}]$.
- The minimum possible sum for A, where $A \in \mathcal{P}(X_n)$, is denoted as $minSum(n)$.

(5) $midSum(n)$ is the mid point of the range of $Sum(A)$ where $A \in \mathcal{P}(X_n)$. Since, the maximum possible sum for power sets of X_n, $\mathcal{P}(X_n)$ is $\frac{n(n+1)}{2}$ and minimum possible sum is 0, $midSum(n) = \frac{minSum(n)+maxSum(n)}{2} = \frac{0+\frac{n(n+1)}{2}}{2}$

$midSum = d = \frac{(1+2+3\ldots n)}{2} = \frac{n(n+1)}{4}$

For simpler calculations, we consider $midSum$ as the largest integer less than or equal to the mid point.

(6) $maxLength(n)$ is the count of all elements of the universal set X_n. This is the maximum possible length for any element of $\mathcal{P}(X_n)$.

- Therefore, $maxLength(n)$ is equal to the cardinality of set X_n.
- $maxLength(n) = |X_n| = |\{1, 2 \ldots n\}| = n$

(7) $Len(A)$ is the number of elements of a set A where A belongs to power sets of X_n, $A \in \mathcal{P}(X_n)$.

- The range of $Len(A)$ is from 1 to n, $Len(A) \in \{1, 2, \ldots, n\}$.
- We consider, count of all elements of subset ϕ as 1. $Len(\phi) = 1$.
- Therefore, the range of $Len(A)$ is from 1 to n. $Len(A) \in [1, n]$.
- The minimum possible length for A, where $A \in \mathcal{P}(X_n)$, is denoted as $minLen(n)$.

(8) $minSum(n, l)$ is the sum of a subset A where $A \in \mathcal{P}(X_n)$ with $Len(A) = l$. A is the subset of length l with minimum possible sum. Subset of length l with minimum possible sum contains first l smallest natural numbers. Therefore, minimum possible subset of length l is $A = \{1, 2 \ldots l\}$.

$minSum(n, l) = (1 + 2 + \ldots + l) = \frac{l(l+1)}{2}$

(9) $maxSum(n, l)$ is the sum of a subset A where $A \in \mathcal{P}(X_n)$ and $Len(A) = l$. A is the subset of length l with maximum possible sum. Subset of length l with maximum possible sum will contain l largest natural numbers decreasing from n.

- Maximum possible subset of length l is A, $A = \{n, n-1 \ldots n-(l-1)\}$.
- $maxSum(n, l) = (n + (n-1) + \ldots + n - l + 1) = n \times l - \frac{l-1(l-1+1)}{2}$

- $maxSum(n,l) = \frac{l(2n-l+1)}{2}$

4 DISTRIBUTION FORMULAE

We have analyzed the distribution of $\mathcal{P}(X_n)$ over sum, length and count of individual elements. We present distribution formulas and algorithms, along with example, which show definite patterns and relations among these subsets.

In table 1, we briefly present the formula, definition, meaning, values and assumptions of all distributions which are required for design and evaluation of alternate enumeration techniques for SSP. Cardinality of a set is the number of elements of the set. These distributions are prepossessing procedures which are required for presenting our novel alternate enumeration techniques for solving SSP. The formulae are the notation developed in Section-3. In Table 1, b denotes the maximum possible sum for any element of $\mathcal{P}(X_n)$, $b = \frac{n(n+1)}{2}$.

Distribution	Formula	Meaning	Value/Assumption
SD Sum-Distribution	A 2D matrix with cardinality $n \times b$, where $\|X_n\| = n$ and $b = \frac{n(n+1)}{2}$.	$SD[n][S]$ represents the count of all the subsets belonging to $\mathcal{P}(X_n)$ with sum S. Every row, $SD[n]$, is the sum distribution for all subsets of X_n where sum is S.	In this paper, the empty set ϕ is counted once while calculating the sum distribution, $SD[n][0] = 1$.
LD Length-Sum-Distribution	A 3D matrix of cardinality $n \times b \times n$, where $\|X_n\| = n$ and $b = \frac{n(n+1)}{2}$.	$LD[n][S][l]$ represents the count of all the subsets belonging to $\mathcal{P}(X_n)$ with sum S and length l. Every column of this matrix, $LD[n][S][l']$, where $\forall l' \in [0,n]$, is the length distribution for all subsets of X_n with sum S.	Extending the previous assumptions we get, $LD[n][S][0] = 1$, $\forall S \in [0,b]$ $LD[n][0][l] = 1$, $\forall l \in [1,n]$
ED Element-Distribution	A 3D matrix of cardinality $n \times b \times n$, where $\|X_n\| = n$ and $b = \frac{n(n+1)}{2}$.	$ED[n][S][e]$ represents the count element e in all the subsets belonging to $\mathcal{P}(X_n)$ with sum S. Every row, $ED[n][S]$, is the element distribution for all subsets of X_n with sum S.	In this paper, we assume the count of element-ϕ in all subsets of $\mathcal{P}(X_n)$ as 0. $ED[n][S][0] = 0$, $\forall S \in [0,b]$ A zero-sum is achieved only by subset ϕ. $ED[n][0][e] = 0$, $\forall e \in [0,n]$.

Table 1. Formula, definition, meaning, values and assumptions of all distributions which are required for design and evaluation of alternate enumeration techniques for SSP. First column denotes the distribution name, second and third column define the formula, definition and concept behind every distribution and fourth column states all the assumptions.

4.1 Sum Distribution

In sum distribution, also referred as *SD*, we find the number of subsets which sum up to a certain integer S, where $X_n = \{1,2,3 \ldots n\}$ and $S \in [0, \frac{n(n+1)}{2}]$. It is represented as $SD[n][S]$. Equation 1 establishes the formula for the sum distribution. Before counting the subsets of a particular sum, we initialize the count as zero, $\forall n, S\ SD[n][S] = 0$. Following are the base cases for sum distribution ($SD[n][S]$):

Values of l for $n = 0$	Subset	Sum of the Subset	No. of Subsets / Length Distribution
l=0	$\{\phi\}$	0	1

Values of l for $n = 1$	Subset	Sum of the Subset	No. of Subsets / Length Distribution
l=0	$\{\phi\}$	0	1
l=1	{1}	1	1

Values of l for $n = 2$	Subset	Sum of the Subset	No. of Subsets / Length Distribution
l=0	$\{\phi\}$	0	1
l=1	{1}	1	1
	{2}	2	1
l=2	{1, 2}	3	1

Table 2. Length-Sum Distribution for base cases: X_0, X_1 and X_2. First column presents the possible length values, second and third column presents the corresponding subsets and their sum respectively and the fourth column presents the Length-Sum distribution, $LD[n][S][l]$.

(1) For $n = 0$ and $S = 0$, the corresponding subset is ϕ. Since, zero-sum ($Sum = 0$) can be achieved only with subset ϕ and $Sum(\phi)$ is assumed to be 0, as defined in Section 3, the count of occurrence of ϕ-subset in $P(X_0)$ is taken as 1. Therefore, $SD[0][0] = 1$.
(2) $\forall i \in [1, n]$ and $S = 0$, $SD[i][0] = 1$. Since, zero-sum ($Sum = 0$) can be achieved only with subset ϕ, the count of occurrence of ϕ-subset in $P(X_n)$ is taken as 1. Therefore, $SD[n][0] = 1$.
(3) $SD[i][j] = 0$, if $i < 0$ or $j < 0$.

$$SD[n][S] = \begin{cases} 1 & (S = 0) \text{ or } (n = 1) \\ 0 & (n = 0) \\ SD[n-1][S] & 0 < S < n \\ SD[n-1][S] + SD[n-1][S-n] & n \leq S \leq \lfloor \frac{n(n+1)}{4} \rfloor \\ SD[n][maxSum(n) - S] & \lfloor \frac{n(n+1)}{4} \rfloor < S \leq maxSum(n) = \frac{n(n+1)}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Similar to Element Distribution (Section 4.3), we can give uniqueness and correctness proof of Sum Distribution.

4.2 Length-Sum Distribution

In length-sum distribution, we find the number of subsets of X_n of length l which sum up to S where $S \in [0, maxSum(n)]$, $maxSum(n) = \frac{n(n+1)}{2}$ and $l \in [0, n]$. Table 2 presents the bases cases for Length-Sum Distribution.

$$LD[n][S][l] = \begin{cases} 1 & l = 0 \text{ and } S = 0 \\ LD[n-1][S][l] & 1 \leq l \leq \lfloor \frac{n}{2} \rfloor \text{ and } 0 \leq S < n \\ LD[n-1][S][l] + LD[n-1][S-n][l-1] & 1 \leq l \leq \lfloor \frac{n}{2} \rfloor \text{ and } n \leq S \leq \frac{n(n+1)}{2} \\ LD[n][maxSum(n) - S][n-l] & \lfloor \frac{n}{2} \rfloor < l \leq n \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Similar to Element Distribution (Section 4.3), we can give uniqueness and correctness proof of Length-Sum Distribution.

Values for n=5														
l=0			l=1			l=2			l=3			l=4		
Sum	Subset	**Size**	Sum	Subset	**Size**	Sum	Subset	**Size**	Sum	Subset	**Size**	Sum	Subset	**Size**
0	ϕ	1	1	{1}	1	3	{1, 2}	1	6	{1, 2, 3}	1	10	{1, 2, 3, 4}	1
			2	{2}	1	4	{1, 3}	1	7	{1, 2, 4}	1	11	{1, 2, 3, 5}	1
			3	{3}	1	5	{1, 4}, {2, 3}	2	8	{1, 2, 5}, {1, 3, 4}	2	12	{1, 2, 4, 5}	1
			4	{4}	1	6	{1, 5}, {2, 4}	2	9	{2, 3, 4}, {1, 3, 5}	2	13	{1, 3, 4, 5}	1
			5	{5}	1	7	{2, 5}, {3, 4}	2	10	{2, 3, 5}, {1, 4, 5}	2	14	{2, 3, 4, 5}	1
						8	{3, 5}	1	11	{2, 4, 5}	1			
						9	{3, 6}	1	12	{3, 4, 5}	1			

l=5		
Sum	Subset	**Size**
15	{1, 2, 3, 4, 5}	1

Table 3. Length-Sum Distribution for $\mathcal{P}(X_5)$

4.3 Element Distribution

In Section 4.1, we have explained and explored the concept of Sum Distribution, where we count the number of subsets out of all power set $\mathcal{P}(X_n)$, of X_n which add up to a certain number S. Let us assume, M represents such sets. We study the occurrence of each element from set X_n in set M. e denotes each element of X_n, $\forall e \in [1,n]$, $\forall S \in [0, \frac{n(n+1)}{2}]$, element distribution function, $ED[n][S][e]$, is defined as follows:

$$ED[n][S][e] = \begin{cases} 0 & (n = 0) \text{ or } (S = 0) \text{ or } (e = 0) \\ & \text{or } (0 < S < n \text{ and } e == n) \\ ED[n-1][S][e] & 0 \leq S < n \text{ and } 1 \leq e < n \\ ED[n-1][S][e] + ED[n-1][S-n][e] & n \leq S \leq \frac{n(n-1)}{2} \text{ and } 1 \leq e < n \text{ and } n > 2 \\ SD[n-1][S-n] & n \leq S \leq \frac{n(n+1)}{2} \text{ and } e == n \\ SD[n][S] - ED[n][maxSum - S][e] & \frac{n(n-1)}{2} + 1 \leq S \leq \frac{n(n+1)}{2} \text{ and } 1 \leq e < n \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Element distribution is another prepossessing procedure required for presenting different alternate enumeration techniques especially bucket algorithms introduced in Section 5.4.

Table 4 represents the count of elements in $\{1,2\}$ and $\{1,2,3\}$ in all subsets of X_2 and X_3 respectively which are divided based on their sums. These are the base cases. Similarly, Table 5 represents distribution of elements of X_5 in $\mathcal{P}(X_5)$, where subsets are categorized on the basis of their Sum. Element distributions of X_0 includes the count of element 0 in subset ϕ with $Sum = 0$. We assume $ED[0][0][0] = 0$. For a given n, the count of element 0 in all the subsets is considered as $NULL$ or 0. We are not including 0 in the set of first n natural numbers. This generate $ED[n][S][0] = 0 \; \forall S \in [0, \frac{n(n+1)}{2}]$. Also, for any value of n, a zero-sum is achieved only by subset ϕ which is an empty set. Therefore, $ED[n][0][e] = 0 \; \forall e \in [0,n]$. We consider values of elements distribution for $\mathcal{P}(X_0)$, $\mathcal{P}(X_1)$ and $\mathcal{P}(X_2)$ as seed values. Following are the values:

(1) $ED[0][0][0] = 0$
(2) $ED[1][1][1] = ED[2][1][1] = 1$
(3) $ED[2][2][2] = 1$
(4) $ED[2][3][1] = ED[2][3][2] = 1$

(5) otherwise $ED[i][j][k] = 0$

Subsets →	ϕ	{1}	{2}	{1, 2}
Elements ↓				
1	0	1	0	1
2	0	0	1	1

Subsets →	ϕ	{1}	{2}	{3}	{1, 2}	{1, 3}	{2, 3}	{1, 2, 3}
Elements ↓								
1	0	1	0	0	1	1	0	1
2	0	0	1	0	1	0	1	1
3	0	0	0	1	0	1	1	1

Table 4. Distribution of elements [1,2] in $\mathcal{P}(X_2)$ and elements [1,2,3] in $\mathcal{P}(X_3)$.

Values for n=5																
No. of Subsets for a Sum	1	1	1	2	2	3	3	3	3	3	3	2	2	1	1	1
Sum →	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Integers ↓																
1	0	1	0	1	1	1	2	1	2	1	2	1	1	1	0	1
2	0	0	1	1	0	1	2	2	1	1	2	2	1	0	1	1
3	0	0	0	1	1	1	1	1	2	2	2	1	1	1	1	1
4	0	0	0	0	1	1	1	2	1	2	2	1	2	1	1	1
5	0	0	0	0	0	1	1	1	2	2	2	2	2	1	1	1

Table 5. Distribution of elements $[1, 2, 3, 4, 5]$ in $\mathcal{P}(X_5)$.

4.3.1 Correctness of the Element Distribution Formula. We present the theorems and lemma which prove the correctness of Element distribution formula, $ED[n][S][e]$ presented in Equation 3. $ED[n][S][e]$ represents the count of element e in those subsets of X_n which has sum S where $e \in [1, n]$, $S \in [0, maxSum]$ and $maxSum = \frac{n(n+1)}{2}$.

Theorem 1. *$ED[n][S][n] = 0$ if $0 < S < n$.*

Proof. Let us assume $ED[n][S][e] \neq 0$ and $ED[n][S][e] = c$, where c is a positive integer. c is the count of number of times an element e occur in a class of subsets $element_{(n,S,e)}$ where $element_{(n,S,e)}$ consist of all the subsets of $\mathcal{P}(X_n)$ which add up to a sum of S. Since, c represents a count, it cannot be negative. By definition c, $ED[n][S][e]$ and $element_{(n,S,e)}$ follow these equations:

$$c = |element_{(n,S,e)}| \tag{4}$$

$$ED[n][S][e] = |element_{(n,S,e)}| \tag{5}$$

$$c = ED[n][S][e] \tag{6}$$

Let A be a subset of $element_{(n,S,e)}$. Then, e will belong to A and sum of all elements of A will be greater than or equal to e.

$$e \in A \tag{7}$$

$$Sum(A) \geq e \tag{8}$$

$$S \geq e \tag{9}$$

Since $(e == n)$ as per the initial conditions, Equation 9 will become,

$$S \geq n \tag{10}$$

Since $0 \leq S < n$ it results into a contradiction. Our assumption is false. There are no subsets which contain e and have sum less than e. Therefore, from the condition $c = 0$ and from Equation 9

$$ED[n][S][e] = 0 \tag{11}$$

$$ED[n][S][e] = 0 \; if \; 0 < S < n \; and \; e == n \tag{12}$$

Hence, we have proved the first part of Equation 3. □

Theorem 2. $ED[n][S][e] = ED[n-1][S][e] + ED[n-1][S-n][e]$ *if* $n \leq S \leq \frac{n(n-1)}{2}$, $1 \leq e < n$ *and* $n > 2$.

Proof. Let $element_{(n,S,e)}$ be a class of subsets which consists of all the subsets of $P(X_n)$ which sum upto S and contain an element e, $1 \leq e < n$. Let us assume, a set $A \in element_{(n,S,e)}$. Since $(S \geq n)$, then A may or may not contain element n. If $n \in A$ then $A - n$ belongs to the class of subsets of $\mathcal{P}(X_{n-1})$ which sum upto $(S - n)$ and contain an element e (as presented in Equation 13). If $n \notin A$ then, A belongs to the class of subsets of $\mathcal{P}(X_{n-1})$ which sum upto S and contain an element e (as presented in Equation 14).

$$A - n \in element_{(n-1,S-n,e)} \tag{13}$$

$$A \in element_{(n-1,S,e)} \tag{14}$$

From Equation 13 and Equation 14, we form the set of all subsets which sum up to S and contain element e,

$$element_{(n,S,e)} = element_{(n-1,S,e)} \cup element_{(n-1,S-n,e)} \quad n \leq S \leq \frac{n(n-1)}{2} \quad and \quad 1 \leq e < n \tag{15}$$

Taking cardinality on both sides of Equation 15,

$$|element_{(n,S,e)}| = |element_{(n-1,S,e)}| + |element_{(n-1,S-n,e)}| \quad n \leq S \leq \frac{n(n-1)}{2} \quad and \quad 1 \leq e < n \tag{16}$$

$$ED[n][S][e] = ED[n-1][S][e] + ED[n-1][S-n][e] \quad n \leq S \leq \frac{n(n-1)}{2} \quad and \quad 1 \leq e < n \tag{17}$$

In order to complete this proof following properties of $element_{(n,S,e)}$ should be proved.

(1) *Uniqueness:* There should be no duplicate subsets in $element_{(n,S,e)}$, $element_{(n-1,S,e)} \cap element_{(n-1,S-n,e)} = \phi$.

Proof. $element_{(n-1,S,e)}$ is the set of all the subsets of $\mathcal{P}\left(X_{(n-1)}\right)$ containing element e with sum S and $element_{(n-1,S-n,e)}$ is the set of all the subsets of $\mathcal{P}\left(X_{(n-1)}\right)$ containing element e with sum $(S - n)$. We use the method of contradiction to prove set of subsets in $element_{(n-1,S,e)}$ and $element_{(n-1,S-n,e)}$ are independent. Let us assume, subset p belongs to both $element_{(n-1,S,e)}$ and $element_{(n-1,S-n,e)}$. Since, $p \in element_{(n-1,S,e)}$, therefore by definition, the subset p contains element e, has elements ranging from 1 to $(n-1)$ and these elements sum upto S.

$$S = \sum_{i=1}^{len(p)} p_i \tag{18}$$

Similarly, as per assumption, $p \in element_{(n-1,S-n,e)}$. Therefore by definition, the subset p contains element e, has elements ranging from 1 to $(n-1)$ and these elements sum upto $(S - n)$.

$$(S - n) = \sum_{i=1}^{len(p)} p_i \tag{19}$$

From Equation 18 and Equation 19, there is a contradiction as $\sum_{i=1}^{len(p)} p_i$ is both S and $(S - n)$. Since, n is a natural number, the above equations contradict our assumption that a subset p can belong to both sets

$element_{(n-1,S,e)}$ and $element_{(n-1,S-n,e)}$. Therefore, by contradiction, there is no subsets p which belongs to both sets. Hence, $element_{(n-1,S,e)}$ and $element_{(n-1,S-n,e)}$ are independent. □

(2) *Completeness:* $element_{(n,S,e)}$ should contain all the subsets of $\mathcal{P}(X_n)$ which contain element e and sum upto S.

Proof. The power set of X_n, $\mathcal{P}(X_n)$ which contain element e and sum upto S can be divided into two parts: subsets with sum S which contain element n and subsets with sum S which do not contain element n. By definition, $element_{(n-1,S,e)}$ is the set of all the subsets of $\mathcal{P}\left(X_{(n-1)}\right)$ with sum S containing element e and $element_{(n-1,S-n,l-1)}$ is the set of all the subsets of $\mathcal{P}\left(X_{(n-1)}\right)$ with sum $(S-n)$ containing element e.

In Equation 15, the union of sets $element_{(n-1,S,e)}$ and $element_{(n-1,S-n,e)}$ generates all subsets of $\mathcal{P}(X_n)$ with sum S containing element e. Therefore, $element_{(n,S,e)}$ should consists of subsets of $\mathcal{P}(X_n)$ with sum S containing element e. □

The above two proofs are required to complete the statement: $ED[n][S][e] = ED[n-1][S][e]+ED[n-1][S-n][e]$ if $n \leq S \leq \frac{n(n-1)}{2}$ and $1 \leq e < n$. This theorem will only be true, if sum is positive i.e. $S \geq 0$

$$S \geq 0 \tag{20}$$

$$\frac{n(n-1)}{2} - n \geq 0 \tag{21}$$

$$\frac{n^2 - n - 2n}{2} \geq 0 \tag{22}$$

$$\frac{n^2 - 3n}{2} \geq 0 \tag{23}$$

$$\frac{n(n-3)}{2} \geq 0 \tag{24}$$

$$n(n-3) \geq 0 \tag{25}$$

Therefore, either both n and $n-3$ should be greater than 0 or both should be less than 0. Since, n cannot be negative,

$$n \geq 0 \quad and \quad n \geq 3 \tag{26}$$

Therefore,

$$n \geq 3 \tag{27}$$

Hence, from Equation 17 and Equation 27 we have proved the third part of Equation 3. □

Lemma 3. $ED[n][S][e] = ED[n-1][S][e]$ *if* $0 \leq S < n$ *and* $1 \leq e < n$.

Proof. According to Theorem 2,

$$ED[n][S][e] = ED[n-1][S][e] + ED[n-1][S-n][e] \quad n \leq S \leq \frac{n(n-1)}{2} \quad and \quad 1 \leq e < n \tag{28}$$

Since,

$$0 \leq S < n \tag{29}$$

$$(-n) \leq S - n < 0 \tag{30}$$

But a sum cannot be negative. Therefore, count of element e in subsets of $\mathcal{P}(X_{n-1})$ which sum up to S is zero, $ED[n-1][S-n][e] = 0$.

$$ED[n][S][e] = ED[n-1][S][e] + 0 \tag{31}$$

$$ED[n][S][e] = ED[n-1][S][e] \quad 0 \leq S < n \quad and \quad 1 \leq e < n \tag{32}$$

Equation 32 proves the second part of Equation 3. □

THEOREM 4. $ED[n][S][e] = SD[n-1][S-n]$ *if* $n \leq S \leq \frac{n(n+1)}{2}$ *and* $e == n$.

PROOF. Let $element_{(n,S,e)}$ be a class of subsets where it consist of all the subsets of $\mathcal{P}(X_n)$ which sum up to S and contain an element e, $e == n$. Let us assume $A \in element_{(n,S,e)}$ and $|element_{(n,S,e)}| = ED[n][S][e] = c$ where $c \geq 0$. Since, element e belongs to set A, $e \in A$,

$$A - e \equiv A - n \in element_{(n-1,S-e,0)} \tag{33}$$

Sum S will result in following condition,

$$n \leq S \leq \frac{n(n+1)}{2} \tag{34}$$

$$0 \leq S - n \leq \frac{n(n+1)}{2} - n \tag{35}$$

Let us assume $S - n$ as S',

$$0 \leq S' \leq \frac{n^2 - n}{2} \tag{36}$$

$$0 \leq S' \leq \frac{n(n-1)}{2} \tag{37}$$

$$maxSum(n-1) = \frac{n(n-1)}{2} \tag{38}$$

From Equation 33 and Equation 38,

$$\forall A - n \in element_{(n-1,S-n,0)} \equiv element_{(n-1,S-n,e')} \quad where \quad 1 \leq e' < n \tag{39}$$

$$\forall A \in element_{(n,S-n+n,n)} \equiv element_{(n-1,S-n,e')} \quad where \quad 1 \leq e' < n \tag{40}$$

$$\forall A \in element_{(n,S,n)} \equiv element_{(n-1,S-n,e')} \quad where \quad 1 \leq e' < n \tag{41}$$

Taking cardinality on both sides,

$$|element_{(n,S,n)}| = |element_{(n,S,r)}| = |element_{(n-1,S-n,e')}| \tag{42}$$

$$ED[n][S][e == n] = ED[n-1][S-n][e'] \tag{43}$$

$|element_{(n-1,S-n,e')}|$ is the number of subsets X_{n-1} which sum up to $(S-n) = (S-e) = (S-n)$. By using the concept of sum distribution defined in Section 4.1 and Equation 43,

$$|element_{(n-1,S',e')}| = SD[n-1][S-n] \tag{44}$$

$$ED[n-1][S'][e'] = ED[n][S][e == n] = SD[n-1][S-n] \tag{45}$$

$$ED[n][S][e] = SD[n-1][S-n] \quad where \quad n \leq S \leq \frac{n(n+1)}{2} \quad and \quad e == n \tag{46}$$

Equation 46 proves the fourth part of Equation 3. □

THEOREM 5. $ED[n][S][e] = SD[n][S] - ED[n][maxSum(n) - S][e]$ *if* $(\frac{n(n-1)}{2} + 1) \leq S \leq \frac{n(n+1)}{2}$ *and* $1 \leq e < n$

Proof. $maxSum(n)$ is the sum of all elements of $X_n = 1 + 2 + \ldots n = \frac{n(n+1)}{2}$, as defined in Section 3. Let us assume $S' = maxSum(n) - S$. Since, the plot between number of subsets and sum follow a Gaussian symmetric distribution, $SD[n][S]$ will be equal to $SD[n][maxSum(n) - S]$.

$$SD[n][S] = SD[n][S'] = c \tag{47}$$

There are c number of subsets which sum up to S and S'. In this case, sum S is greater than the $\frac{maxSum}{2}$ (the mid point) and by using the reflection/symmetric property of the curve we can find all the values of $ED[n][S][e]$.

$$S'' = \frac{maxSum}{2} = \frac{n(n+1)}{4} \tag{48}$$

$$S_{low} = \frac{n(n-1)}{2} + 1 \tag{49}$$

$$S_{low} - S'' = \frac{n(n-1)}{2} + 1 - \frac{n(n+1)}{4} \tag{50}$$

$$S_{low} - S'' = \frac{n(2n-2-n+1)}{2} + 1 \tag{51}$$

$$f(n) = S_{low} - S'' = \frac{n^2 - 3n + 2}{2} \tag{52}$$

By using the property of second derivative test we show that $f'(n)$ is greater than 0 when $S > \frac{maxSum}{2}$.

$$f'(n) = \frac{d(f(n))}{dn} > 0 \tag{53}$$

$$f'(n) = \frac{d(n^2 - 3n + 2/2)}{dn} > 0 \tag{54}$$

$$f'(n) = n - \frac{3}{2} > 0 \tag{55}$$

$$f'(n) = n > \frac{3}{2} \tag{56}$$

Therefore, $\forall n \geq 2$ we can use the symmetric property and calculate half of the values by using the previously calculated values. For $n = 1$ values of element distribution will be covered as the part of base cases.
Let $sum_{(n,S)}$ be a set of all the subsets of $\mathcal{P}(X_n)$ which sum up to S and $sum_{(n,S')}$ consist of all subsets of $\mathcal{P}(X_n)$ which sum to S', where $S' = (maxSum - S)$. $\forall A \in sum_{(n,S)}$ and $A^c \in sum_{(n,S')}$ where A^c is the complement set of A.

$$A \cup A^c = U \tag{57}$$

Since, U is the universal set, $U = \{1, 2 \ldots n\}$ and contain a single occurrence of each element $e \in [1, n]$, therefore, $A \cup A^c$ also contains a single occurrence of each element e. From Equation 57 there are c subsets in A and A^c. $\forall k \in [1, n]$ count of e in A and A^c is 1. Let us define $Count(x, y)$ as the count of element x in any subset or class of subsets y.

$$Count(e, A) + Count(e, A^c) = 1 \tag{58}$$

$\forall e \in [1, n], \forall A \in sum_{(n,S)}$ *and* $\forall A^c \in sum_{(n,S')}$

$$Count(e, sum_{(n,S)}) + Count(e, sum_{(n,S')}) = |sum_{(n,S)}| * 1 = |sum_{(n,S')}| * 1 \tag{59}$$

By using the definition of element distribution and Equation 47

$$ED[n][S][e] + ED[n][S'][e] = c \tag{60}$$

$$ED[n][S][e] + ED[n][S'][e] = SD[n][S] \tag{61}$$

$$ED[n][S][e] = SD[n][S] - ED[n][S'][e] \tag{62}$$

Therefore, by putting the value of $S' = (maxSum(n) - S)$

$$ED[n][S][e] = SD[n][S] - ED[n][maxSum(n) - S][e] \tag{63}$$

Equation 63 proves the last part of Equation 3. □

Sum Distribution, Length-Sum Distribution and Element Distribution are used in developing alternate enumeration techniques for solving SSP. These techniques are presented in the next section.

5 ALTERNATE ENUMERATION TECHNIQUES FOR SUBSET SUM PROBLEM

In this paper, we propose seven approaches to find the solution for enumerating all the $(2^n - 1)$ subsets of X_n. In each approach, we choose different method for addressing the enumeration of SSP. We propose novel algorithms: Subset Generation using Sum Distribution (SDG), Subset Generation using Length-Sum Distribution (LDG), Basic Bucket Algorithm (Basic BA), Maximum and Minimum Frequency Driven Bucket Algorithms (Max FD and Min FD) and Local Search using Maximal and Minimal Subsets (LS MaxS and LS MinS) for enumerating SSP. The first approach is the backtracking algorithm. It is the naive method for solving SSP. This algorithm is used to benchmark the new proposed algorithms.

5.1 Subset Generation using Backtracking

Our aim is to find all the subsets of set X_n with $Sum = S$. According to the exhaustive search algorithm for SSP [19], we try to find the resulting subset by iterating through all possible 2^n solutions. But in this algorithm, we arrange the elements in an orderly fashion. The worst case time complexity for this algorithm is $O(n \times 2^n)$. The space complexity for this algorithm is the size of the input, $O(n)$. Even though backtracking is a clean and crisp algorithm for SSP, this algorithm has many drawbacks. It tries to generate all the desired subsets by checking every branch and subset. Since there can be many branches at every state of the back tracking algorithm, this leads to inefficient multiple recursive calls and reversion to old states. It requires a large amount of time and space to reflect the changes in the system stack.

5.2 Subset Generation using Sum Distribution

We design a generator using Sum Distribution. Algorithm 1 is the pseudo-code for generating all the subsets of X_n with sum S. As we know, sum distribution is recursive and uses subsets of $X_{(n-1)}$ to produce results for X_n. We store these previous values with the help of SDG (initialized at Line 1). Extra values of SDG ($SDG[i-1]$) are freed in Line 20 to minimize the space consumption. In Line 2, we iterate through smaller natural numbers. Line 3 to Line 6 define $start\_sum$, mid_sum, end_sum and $universal\_set$. Line 7 to Line 19 iterate through values of sum between $start\_sum$ and mid_sum. The desired set of subsets, $SDG[i][j]$ (subsets of X_i with sum j), consists of all subsets of $SDG[i-1][j]$ and $SDG[i-1][j-i]$. Next, we include i^{th} element in every subset of $SDG[i-1][j-i]$. For each of these resulting subsets, a symmetric subset of sum $(end\_sum - j)$ is calculated by subtracting the subset from $universal\_set$. Line 11 to Line 18 essentially execute these steps and returns the final result at Line 22.

The value of maximum number of subsets has exponential bound, $O(2^n * n^{\frac{-3}{2}})$, as described in Appendix B. Therefore, the time complexity for $(loop_3)$ at Line 14 is $O(2^n * n^{\frac{-3}{2}})$. Since, $n \in [1, n]$ and $S \in [0, \frac{n(n+1)}{2}]$, time complexity of the above algorithm results to $O(loop_1) * O(loop_2) * O(loop_3) = O(n) * O(n^2) * O(2^n * n^{\frac{-3}{2}}) = O(2^n * n^{\frac{3}{2}})$. Space complexity for the above algorithm is the size of array storing smaller subsets, $SDG[n-1][S]$. This complexity is also exponential $n * S * \ No.\ of\ Subsets$. Since $S \in [0, \frac{n(n+1)}{2}]$, the space complexity results to $O(n) * O(n^2) * O(2^n * n^{\frac{-3}{2}})$ i.e. $O(2^n * n^{\frac{3}{2}})$.

Algorithm 1 SDG: GeneratorUsingSumDistribution(n)

1: $SDG = \{\}$ ▷ Data structure to store the generated Subsets
2: **for** $i \in \{1, \ldots, n\}$ **do**
3: $start\_sum = 0$
4: $mid\_sum = \lfloor \frac{i(i+1)}{4} \rfloor$
5: $end\_sum = \frac{i(i+1)}{2}$ ▷ end_sum is equal to $maxSum(i)$
6: $universal\_set = \{1, 2 \ldots n\}$ ▷ $universal\_set$ is used to calculate the symmetric subsets
7: **for** $j \in \{start\_sum, \ldots, mid\_sum\}$ **do**
8: **if** $(j == 0)$ **then**
9: $SDG[i] = \{\phi\}$
10: **end if**
11: $SDG[i][j] = SDG[i-1][j]$
12: **for** $subset \in SDG[i-1][j-i]$ **do**
13: $subset.append(i)$
14: $SDG[i][j].append(subset)$ ▷ Adding i^{th} element in every subset of $SDG[i-1][j-i]$
15: **end for**
16: **if** $j \neq (i-j)$ **then**
17: $SDG[i][end\_sum - j] = universal\_set - SDG[i][j]$ ▷ Symmetric subsets.
18: **end if**
19: **end for**
20: $Free(SDG[i-1])$
21: **end for**
22: **return** $SDG[n]$

5.3 Subset Generation using Length-Sum Distribution

Along with Sum Distribution, we have established several concepts, theories and formulas for $Length - Sum$ Distribution as well. It counts the number of subsets of X_n of length l and sum S where $X_n = \{1, 2, 3 \ldots n\}$, $l \in [0, n]$ and $S \in [0, \frac{n(n+1)}{2}]$, represented by $LD[n][S][l]$. The recursive equation (Equation 2) establishes the theory for the Length-Sum distribution.

In this section, we present the designed generator. Algorithm 2 is the pseudo-code for generating all the subsets of X_n of length l and sum S. This distribution is recursive and uses LDG to store the previous output which is initialized at Line 1 and Line 10. The notation for LDG is different than notation of LD. We denote the count the number of subsets of X_n of length l and sum S where $X_n = \{1, 2, 3 \ldots n\}$, $l \in [0, n]$ and $S \in [0, \frac{n(n+1)}{2}]$ by $LD[n][S][l]$. However, $LDG[i][j][k]$ consists of all subsets of X_i with $length = j$ and $Sum = k$. In LDG, notation for length and sum are reversed for easier calculations.

In Algorithm 2, extra values of LDG ($LDG[i-1]$) are freed in Line 26 to minimize the space consumption. In Line 5, Line 9 and Line 13, we iterate through smaller natural numbers, length range and possible values of sum respectively. Line 6 to Line 12 we define max_sum for X_i, bases cases of $LDG[i][j]$, $start\_sum$ and end_sum. Line 13 to Line 24 iterates through feasible values of sum between $start\_sum$ and end_sum. The desired set of subsets, $LDG[i][j][k]$ consists of all subsets of $LDG[i-1][j][k]$ and $LDG[i-1][j-1][k-i]$. We include i^{th} element in every subset of $LDG[i-1][j-1][k-i]$. For each of these resulting subsets, a symmetric subset of length $(i-j)$ and sum $(end\_sum - k)$ is calculated by subtracting the subset from $universal\_set$. Line 15 to Line 23 essentially execute these steps and returns the final result at Line 28.

The value of maximum number of subsets has exponential bound, $O(2^n * n^{\frac{-3}{2}})$, as described in Appendix B. Therefore, the time complexity for ($loop_4$) in Line 16 is $O(2^n * n^{\frac{-3}{2}})$. Since, $l \in [1, n]$ and $S \in [0, \frac{n(n+1)}{2}]$ time complexity of the above algorithm results to $O(loop_1) * O(loop_2) * O(loop_3) * O(loop_4)$ i.e. $O(n) * O(n) * O(n^2) * O(2^n * n^{\frac{-3}{2}}) = O(n^4 * 2^n * n^{\frac{-3}{2}}) = O(2^n n^{\frac{5}{2}})$. Space complexity for the above algorithm is the size of array storing smaller subsets, $LDG[n-1]$. This complexity is also exponential $n * l * S *$ $(No.\ of\ Subsets)$. Since, $l \in [1, n]$ and $S \in [0, \frac{n(n+1)}{2}]$ the space complexity results to $O(n) * O(n) * O(n^2) * O(2^n * n^{\frac{-3}{2}}) = O(n^4) * O(2^n * n^{\frac{-3}{2}})$ i.e. $O(2^n * n^{\frac{5}{2}})$.

Algorithm 2 LDG: GeneratorUsingLengthSumDistribution(n)

1: $LDG = \{\}$
2: $LDG[0][0] = LD[1][0] = \{\}$ ▷ Base Cases
3: $LD[1][1] = \{[1]\}$ ▷ Base Cases
4: $universal\_set = [1, 2 \ldots n]$ ▷ $universal\_set$ is used to calculate the symmetric subsets
5: **for** $i \in \{2, \ldots, n\}$ **do**
6: $max\_sum = \frac{i(i+1)}{2}$
7: $LDG[i][0] = \{[1]\}$
8: $LDG[i][max\_sum] = \{universal\_set\}$
9: **for** $j \in \{1, \ldots, \frac{i}{2}\}$ **do** ▷ Iterarting till mid point
10: $LDG[i][j] = \{\}$
11: $start\_sum = \frac{j(j+1)}{2}$
12: $end\_sum = i * j - \frac{i(i-1)}{2}$
13: **for** $k \in \{start\_sum, \ldots, end\_sum\}$ **do**
14: $LDG[i][j][k] = LDG[i-1][j][k]$
15: **if** $j \geq 1$ and $k \geq i$ and $i \leq k \leq \frac{i(i+1)}{2}$ **then**
16: **for** $subset \in LDG[i][j-1][k-i]$ **do**
17: $subset.append(i)$
18: $LDG[i][j][k].append(subset)$ ▷ Adding i^{th} element in every subset of $LDG[i-1][j-1][k-i]$
19: **end for**
20: **end if**
21: **if** $j \neq (i-j)$ **then**
22: $LDG[i][i-j][end\_sum - k] = universal\_set - LDG[i][j][k]$ ▷ Symmetric subsets
23: **end if**
24: **end for**
25: **end for**
26: $Free(LDG[i-1])$
27: **end for**
28: $Return\ LD[n]$

5.4 Subset Generation using Basic Bucket Algorithm

In this section, we present a new method which generate all the subsets of X_n with a particular sum. This is a greedy algorithm. The look-up table that has been used, has been explained in Section A. It has been extensively used with this algorithm.

The core idea behind this enumeration technique is to use the various distribution values that we have calculated so far, to construct all the subsets of X_n which sum up to S.

Given: The first concept used for Basic Bucket Algorithm is Element Distribution. We start with the exact occurrence of each element of X_n in subsets of precise sum, S. This information is denoted by $ED[n][S][e]$. The next concept used is the number of subsets, among power set of X_n, where summation of all elements is S. $SD[n][S]$ denotes such count. For this algorithm, we consider $SD[n][S]$ as number of empty buckets. Buckets are storage data structures which are used to stack all the appropriate elements that compute the total sum S. We iterate through all elements in descending order. During each iteration, an element is assigned to one of the buckets. This method is about adding the correct element to the corresponding subset.

Properties: Element distribution and below properties help us to ensure the correct placement for every element.

(1) An element e is added to a bucket b only if the addition results to the uniqueness among all existing elements of the bucket b. This property is followed to guarantee that the generated result is a subset and it is not a bag. A subset belongs to power sets of X_n $\mathcal{P}$ (X_n).
(2) An element e is added to a bucket b only if the addition of the element results to uniqueness amongst all the buckets. We follow this property to ensure the generation of correct number of subsets of sum S.
(3) An element e is added to a bucket b only if on adding the new element, the sum of the bucket does not exceed the desired sum S. This property allows us to create subsets of sum S.

Unfortunately, we have no rule which forces only the generation of subsets with sum S. Many subsets with sum less than S are generated during the first iteration of this technique. These subsets are called the *undesired* set. For every subset of the *undesired* set, A, $Sum(A)$ is less than S i.e. $Sum(A) < S$. Therefore, we have converted this technique to a greedy algorithm. Instead of using this as a one time procedure, we reapply it with modified values of element distribution, $ED[n][S][e]$ and sum distribution, $SD[n][S]$. All subsets are generated by applying the same technique on modified input in a greedy manner.

Uniqueness: The key step in successfully generating the full desired results is to maintain an efficient and complete lookup table as described in Section A. This lookup table which is maintained with the help of a hash function and bit vectors, not only ensures uniqueness among and within the buckets but also makes sure that all the *undesired* subsets of previous iterations are properly hashed. So, we do not regenerate the same *undesired* set in the next iteration. We need to put extra effort to preserve the state of all *undesired* sets from every iteration. The whole lookup table is no bigger than 2^n and every subset: desired or undesired, is stored in the form of one integer *num*, where $num \in [0, 2^n]$. With the aim of preserving the count of every element from the set X_n, we maintain a log table for each round of iterations. The value of log table for each element, at the start of every round is the summation of value of element distribution at the end of last iteration of previous round and the count of all these elements from buckets which do not provide a subset of desired sum.

Algorithm 3 calculates the element distribution before start of each round of Basic Bucket Algorithm. Algorithm 4 initializes the buckets at the start of the algorithm. It finds the value of x and accordingly fill the buckets with the starting elements. This method is called from Line 3 of the function GeneratingSubsets($n, S, SD[n][S], ED[n][S], prevWrongSubsets$) from the main Algorithm 6. We find an appropriate bucket for every element based on the properties of the Basic Bucket Algorithm. Functionality is defined in Algorithm 5. While Algorithm 7 iterates though all the rounds of the bucket algorithm. All iterations of every round is implemented by the Algorithm 6.

For a given n and S, time complexity of the algorithm depends on the maximum number of subsets and time to find a bucket for each element placement. Since, finding the bucket is an iterative algorithm, time taken for this submethod is also proportional to the number of subsets, $SD[n][S]$. Since, the value of maximum number of subsets has exponential bound, $O(2^n * n^{\frac{-3}{2}})$, as described in Appendix B, time complexity is $O(max(SD[n][S] \cdot max(SD[n][S]) =$

Algorithm 3 Basic BA: GetED($n, S, Table, wrongSubsets$)

```
1: function GetED(n, S, Table, wrongSubsets)
2:     newTable = Table
3:     for subset ∈ wrongSubsets do
4:         for ele ∈ subset do
5:             newTable[ele]+ = 1      ▷ Restoring all the ellments of wrongSubsets to the element distribution
6:         end for
7:     end for
8:     Return newTable
9: end function
```

Algorithm 4 Basic BA: InitializeBuckets($all\_buckets$, $Table$, n, S , p)

```
1: function InitializeBuckets(all_buckets, Table, n, S , p)
2:     q = count of non-zero entries of Table
3:     x = min(p, q)
4:     elements = x largest integers of X_n where Table[ele] ≠ 0 ∀ ele ∈ elements
5:     Sort elements in descending order
6:     bucketIndex = 1
7:     for ele in elements do
8:         Add ele in all_buckets[bucketIndex]
9:         bucketIndex + +
10:    end for
11: end function
```

Algorithm 5 Basic BA: FindBucket($all\_buckets$, ele, S)

```
1: function FindBucket(all_buckets, ele, S)
2:     for bucket in all_buckets do
3:         if any bucket entry is same as ele then
4:             Next
5:         else if on adding ele in bucket, Sum(bucket) > S then
6:             Next
7:         else if on adding ele in bucket, bucket becomes duplicate to any other bucket then
8:             Next
9:         else
10:            Return bucket
11:        end if
12:    end for
13:    Return False
14: end function
```

$O(2^n * n^{\frac{-3}{2}} \cdot 2^n * n^{\frac{-3}{2}}) = O(2^{2n} \cdot n^{-3})$. Therefore, for given n and S, the time complexity to generate all the subsets is $O(2^{2n} \cdot n^{-3})$. Space complexity includes size of two storages $Table$ and $all\_buckets$, $O(n) + O(2^n) = O(2^n)$.

Algorithm 6 Basic BA: Generating Subsets($n, S, SD[n][S], ED[n][S], prevWrongSubsets$)

Given: n, S, $SD[n][S]$ and $ED[n][S][i]$ where $i \in [1, n]$
$desiredSubsets = [\ ]$ ▷ $desiredSubsets$ are all the subsets of X_n with sum S.
$wrongSubsets = [\ ]$ ▷ $wrongSubsets$ are the set of *undesired* and *smaller*subsets.
$Table = GetED(n, S, ED[n][S], prevWrongSubsets)$ ▷ the count of every element in subsets of X_n with sum S called from function $GetED$ described in Algorithm 3
$p = SD[n][S]$: number of subsets of X_n with sum S

function GenerateSubsets
    $all\_buckets = p$ empty buckets
    initializeBuckets($all\_buckets$, $Table$, n, S, p) ▷ Initial Step
    $fillBuckets = True$ ▷ Flag to control implementation of the while loop
    while ($fillBuckets$ is set & ($|all\_buckets| > 0$)) **do**
        $filBuckets = False$
        q = count of non-zero entries of $Table$
        $x = min(p, q)$
        $elements = x$ largest integers of X_n where $Table[ele] \neq 0\ \forall\ ele \in elements$
        Sort $elements$ in descending order
        for $ele \in elements$ **do**
            b = findBucket($all\_buckets$, ele, S)
            if b is a bucket **then** ▷ When an elemnt can be inserted in a valid bucket.
                Add ele in bucket b
                $fillBuckets = True$ ▷ If no element is alloted to any bucket in a full iteration.
                $Table[ele] - -$
                if Sum of the bucket $b == S$ **then**
                    $desiredSubsets+ = b$
                    print bucket b
                    Remove b from $all\_buckets$
                    $|all\_buckets| - -$
                end if
            end if
        end for
    end while
    for $bucket \in remaining\_buckets$ **do**
        $wrongSubsets+ = bucket$
    end for
    Return $wrongSubsets$, $Table$
end function

5.5 Subset Generation using Frequency Driven Bucket Algorithms

After the basic bucket algorithm, we present two more bucket algorithms. While the previous algorithm uses the direct information provided by element distribution, $ED[n][S][e]$ and sum distribution, $SD[n][S]$, in these two algorithms, we use element distribution in decreasing or increasing order. In other words, instead of assigning elements to a corresponding bucket in descending order, we assign elements to buckets based on their frequencies. Frequency of an element in all the subsets of X_n with sum S, by definition, is equal to the count of the elements,

Algorithm 7 Basic BA: main Function(n, S)

```
1: function mainFunction(n, S)
2:     prevWrongSubsets = [ ]
3:     prevTable = ED[n][S]
4:     countSubsets = SD[n][S]
5:     while countSubsets > 0 do
6:         prevWrongSubsets, prevTable = GeneratingSubsets(n, S, countSubsets,
                                                            prevTable, prevWrongSubsets)
7:         countSubsets = SD[n][S] = |prevWrongSubsets|
             ▷ Count of Subsets to be generated in next round is same as the size of wrong no. of subsets from
   previous round.
8:     end while
9:     Return True
10: end function
```

denoted by $ED[n][S][e]$. These algorithms are called *Frequency-Driven (FD) Bucket* algorithms. These can be called minimum FD or maximum FD bucket algorithms.

Information used by these algorithms is same as the Basic Bucket Algorithm. While the basic bucket algorithm is iterative, the minimum or maximum frequency driven algorithms are recursive. Information required by this algorithm, properties of elements that should be followed and the measures by which we ensure uniqueness (i.e. using log and lookup tables) is same as the primitive algorithm defined in Section 5.4.

Next, we generate all twenty subsets of X_{10} with $Sum = 15$. For both the algorithms, we select an element based on minimum or maximum frequency. In case of Minimum FD bucket algorithm, we select the maximum element with minimum frequency and recursively produce all the subsets of desired sum. For Maximum FD, we select maximum element with maximum frequency. In Table 6, we log all the iterations for generating all twenty subsets of X_{10} with $Sum = 15$. Following points briefly describe the working of Minimum FD bucket algorithm:

(1) By following the algorithm, we select element 10. Since, $ED[10][15][10] = 3$, first iteration generates 3 subsets: $\{\{10,5\},\{10,4,1\},\{10,3,2\}\}$. This is shown in the first row of Table 7. All subsets are generated in eight iterations.
(2) In next three iterations, we choose elements-9, 8 and 7 respectively, to generate next thirteen subsets. This will results in production of sixteen subsets.
(3) In every iteration, we update the count of elements according to the resulting subsets.
(4) In fifth iteration, we select element 2 and recursively generate two subsets: $\{\{2,6,4,3\},\{2,5,4,3,1\}\}$.

For maximum frequency driven bucket algorithm we select the maximum element with maximum frequency in every iteration.

(1) All twenty desired subsets are produced in seven iterations.
(2) Since, $ED[10][15][1] = ED[10][15][2] = 9$ and $max(1,2)$, we select element 2 and generate nine subsets.
(3) In second iteration, we select element 4 and recursively generate next four subsets: $\{9,6\}$, $\{9,5,1\}$, $\{9,4,2\}$ and $\{9,3,2,1\}$.
(4) Table 6 and Table 7 presents the log entries and subsets corresponding to all iterations of maximum FD bucket algorithm for X_{10} with $sum = 15$.

$Elements \rightarrow$ $Iterations \downarrow$	1	2	3	4	5	6	7	8	9	10
0^{th} $iteration$	9	9	8	8	8	6	5	5	4	**3**
1^{st} $iteration$	8	8	7	7	7	6	5	5	**4**	0
2^{nd} $iteration$	6	6	6	6	6	5	5	**5**	0	0
3^{rd} $iteration$	4	4	5	4	5	4	**4**	0	0	0
4^{th} $iteration$	2	**2**	3	3	4	3	0	0	0	0
5^{th} $iteration$	2	1	2	**2**	3	3	0	0	0	0
6^{th} $iteration$	1	0	1	1	2	**2**	0	0	0	0
7^{th} $iteration$	1	0	1	0	1	1	0	0	0	0

$Elements \rightarrow$ $Iterations \downarrow$	1	2	3	4	5	6	7	8	9	10
0^{th} $iteration$	9	**9**	8	8	8	6	5	5	4	3
1^{st} $iteration$	5	0	4	**4**	5	4	3	2	2	2
2^{nd} $iteration$	**3**	0	2	0	4	3	2	2	2	1
3^{rd} $iteration$	0	0	1	0	2	1	2	1	1	**1**
4^{th} $iteration$	0	0	1	0	1	1	2	1	**1**	0
5^{th} $iteration$	0	0	1	0	1	0	2	**1**	0	0
6^{th} $iteration$	0	0	1	0	1	0	**1**	0	0	0
7^{th} $iteration$	0	0	0	0	0	0	0	0	0	0

Table 6. Log table for iterations of Minimum and Maximum Frequency Driven Bucket Algorithm. We are generating all twenty subsets of X_{10} with $Sum = 15$. Every column denotes the frequency calculation for ten elements and every row denotes the frequency calculations in every iteration. In this table the frequency of every selected element in the previous iteration is marked as bold.

$Iterations$	$Selected\ Element$	$Subsets$	$\|Subsets\|$
1^{st} $iteration$	10	{{10, 5}, {10, 4, 1}, {10, 3, 2}}	3
2^{nd} $iteration$	9	{{9, 6}, {9, 5, 1}, {9, 4, 2}, {9, 3, 2, 1}}	4
3^{rd} $iteration$	8	{{8, 7}, {8, 6, 1}, {8, 5, 2}, {8, 3, 4}, {8, 4, 2, 1}}	5
4^{th} $iteration$	7	{{7, 6, 2}, {7, 5, 3}, {7, 5, 2, 1}, {7, 4, 3, 1}}	4
5^{th} $iteration$	2	{{2, 6, 4, 3}, {2, 5, 4, 3, 1}}	2
6^{th} $iteration$	4	{{4, 6, 5}}	1
7^{th} $iteration$	6	{{6, 5, 3, 1}}	1
$Total\ Number\ of\ Subsets \rightarrow$			20

$Iterations$	$Selected\ Element$	$Subsets$	$\|Subsets\|$
1^{st} $iteration$	2	{{2, 1, 3, 4, 5}, {2, 1, 3, 9}, {2, 1, 4, 8}, {2, 1, 5, 7}, {2, 3, 4, 6}, {2, 3, 10}, {2, 4, 9}, {2, 5, 8}, {2, 6, 7}}	9
2^{nd} $iteration$	4	{{4, 1, 3, 7}, {4, 1, 10}, {4, 3, 8}, {4, 5, 6}}	4
3^{rd} $iteration$	1	{{1, 5, 9}, {1, 6, 8}, {1, 3, 5, 6}}	3
4^{th} $iteration$	10	{{10, 5}}	1
5^{th} $iteration$	9	{{9, 6}}	1
6^{th} $iteration$	8	{{8, 7}}	1
7^{th} $iteration$	7	{{7, 5, 3}}	1
$Total\ Number\ of\ Subsets \rightarrow$			20

Table 7. Log table for iterations of Minimum and Maximum Frequency Driven Bucket Algorithm. We are generating all twenty subsets of X_{10} with $Sum = 15$. First column represents all the iterations, second column shows the chosen element as per the frequency. Third column stores the subsets and the fourth column denotes the count of these subsets.

5.6 Algorithm and Complexities

We state the pseudo codes for solving minimum and maximum FD bucket algorithms. Algorithm 8 updates the element distribution after every iteration and is called from Algorithm 10. This update ensures that correct number of subsets are generated. In Line 5, the element count is reduced according to the answer generated so far. The main function which was defined in Algorithm 9, repeatedly calls the *GeneratingSubsetsbyFDBucketAlgo* function and updates following information:

(1) *countSubsets* - No. of subsets left.
(2) $fullTable$ - Element distribution of X_n with $Sum = S$.
(3) *elements* - Remaining elements which form the remaining subsets.

(4) *elementsCovered* - Elements which are not allowed or required to form remaining subsets.

Apart from these helper methods, the main functionality is presented in Algorithm 10. First we define the input for our algorithm. Line 2 is the base case of our recursive algorithm. We terminate the recursion when the desired sum is S or S is less than zero or there are no *elements* left to generate the subsets. In Line 7 and Line 8, we find $minKey$ and $minVal$ pair. In case of Minimum FD algorithm ($minKey, minVal$) is the largest element with minimum frequency, $e \in [1, n]$ where $ED[n][S][e]$ is minimum. For maximum FD algorithm, we find ($maxKey, maxVal$), the largest element with maximum frequency. The pseudo code for both algorithms are similar. Therefore, we described the minimum FD bucket algorithm only. The main idea behind this algorithm is to find $minKey$, and generate subsets of X_n with $Sum = [S - minKey]$. This means by adding $minKey$ to *elementsCovered*, in Line 10, we do not include it in future partial subsets. In Line 11, we recursively call *GeneratingSubsetsbyFDBucketAlgo* function with modified values. The remaining part of the code is divided in two conditions which are based on the return values from Line 11. It can either be empty or non-empty. $minKey$ is appended to every returning subset of $desiredSubsets[S - minKey]$ and *elementsCovered* are also updated accordingly. In last few lines, we increase the count of $ED[n][S][e]$ for next iteration. This step ensures that the correct subsets are created in next iteration as well.

For a given n and S, time complexity of the maximum or minimum FD bucket algorithm depends on the maximum number of subsets and time taken to solve one recursion. Since, iterating through all elements is a recursive algorithm, time taken for this sub-method is also proportional to the number of subsets, $SD[n][S]$. Since the value of maximum number of subsets has exponential bound, $O(2^n * n^{\frac{-3}{2}})$, as described in Appendix B, time complexity is $O(max(SD[n][S] \cdot max(SD[n][S]) = O(2^n * n^{\frac{-3}{2}} \cdot 2^n * n^{\frac{-3}{2}}) = O(2^{2n} \cdot n^{-3})$. Therefore, for given n and S,the time complexity to generate all the subsets is $O(2^{2n} \cdot n^{-3})$. Space complexity includes size of two storages $Table$ and $desiredSubsets$, $O(n) + O(2^n) = O(2^n)$.

Algorithm 8 Max FD: GetED($n, S, Table, desiredSubsets$)

```
1: function GetED(n, S, Table, desiredSubsets)
2:     newTable = Table
3:     for subset ∈ desiredSubsets do
4:         for ele ∈ subset do
5:             newTable[ele]− = 1          ▷ Reducing count of elements according to desiredSubsets.
6:         end for
7:     end for
8:     Return newTable
9: end function
```

5.7 Subset Generation using Local Search

Our next enumeration technique for subset generation is called the Local Search. Before proceeding with this algorithm, we define two new types of subsets called Maximal and Minimal subsets. They act as the starting point for the local search algorithm.

5.7.1 Maximal and Minimal Subsets. We present a new idea to categorize subsets of a given class. First, we divide the power set of X_n, $\mathcal{P}(X_n)$, on the basis of their sum and then further partition these subsets according to their length. We have formulated and explained this selection process in Section 4.2.

For defining maximal subset, we have the set of first n natural numbers, X_n, sum(S) which belongs to $[0, maxSum(n)]$ where $maxSum(n) = \frac{n(n+1)}{2}$ and length(l) which belongs to $[0, n]$. Consider, A denotes the

Algorithm 9 Max FD: main Function(n, S)

```
1:  function mainFunction(n, S)
2:      fullTable = ED[n][S]
3:      countSubsets = SD[n][S]
4:      elements = [1, 2 ... n]                                  ▷ Available elements
5:      elementsCovered = [ ]                                    ▷ Elements covered so far
6:      while countSubsets > 0 do
7:          desiredSubsets = GeneratingSubsets(n, S, countSubsets,
                                    elements, elementsCovered, fullTable)
8:          countSubsets = SD[n][S] − |desiredSubsets|
9:          Update fullTable, elements, elementsCovered
                          ▷ Reduce frequency of elements according to desiredSubsets.
10:     end while
11:     Return True
12: end function
```

subsets of X_n with length l and sum S. We denotes A as $A = \{A_1, A_2 \dots A_k\}$ where $k = LD[n][S][l]$, the total count of subsets with length l and sum S. A_i represents i^{th} subset of set A and $A_{i,j}$ represents j^{th} element of subset A_i where $j \in [1, l]$. There exists a maximal subset of X_n of length l and sum S, $A_{maximal}$, is defined such that $\forall j \in [1, l]\ A_{maximal,j} > A_{p,j}$ where $p \in \{[1, k] - maximal\}$. There also exists a minimal subset, $A_{minimal}$, defined such that $\forall j \in [1, l] A_{minimal,j} > A_{p,j}$ where $p \in \{[1, k] - minimal\}$.

The key point is that not all values of $A_{maximal,j}$ will be greater than j^{th} element of other subsets in A but there will surely be a subset for which first q elements are greater than rest of the subsets, where $q \in [1, l]$. In order to generate the subset $A_{maximal}$ for X_n for a given sum S and length l, we find the smallest possible element for every position, starting from the rightmost position. This pattern of element generation will ensure largest possible elements at the start of the subset, resulting in the maximal subset. Similarly, we find the largest possible element for every position of minimal subset starting from the rightmost position which ensures the smallest possible element at the start of the subset, resulting in the desired minimal subset. Table 8 displays the maximal and minimal subsets for every sum and length pair of X_5.

The core idea for the local search algorithm is to find all possible subsets of a particular length l and sum S where our starting subset can be a maximal or minimal subset. We find subsets by iterating over length between l_{min} and l_{max} where these are the minimum and maximum possible subsets of X_n with sum s respectively. This is a heuristic algorithm. Next, we present a few examples to explain local search using maximal and minimal subset respectively.

Maximal subset has the largest possible element at every position for a given sum S and length l. Therefore, for local search starting with the maximal subset, we begin from left most element, decrement the first permissible element followed by increment of next permissible element. On contrary, minimal subset has smallest possible element at every position for a given sum S and length l. Therefore, we begin from left most element, increment the first permissible element followed by decrement of next permissible element. Every increment or decrement consists of one unit.

(1) Figure 1 shows the local search example for $n = 10$, $sum = 21$ and $length = 3$ where the starting subset is the maximal subset of respective length.
   (a) We start with subset $\{6, 7, 8\}$. By decrementing the first permissible element 6 by 1 and incrementing third permissible element 8 by 1, we generate the second subset $\{5, 7, 9\}$. We cannot increment the

Algorithm 10 Max FD: GeneratingSubsetsbyFDBucketAlgo($n, S, SD[n][S], elements, elementsCovered, ED[n][S]$)

Given: n, S, $SD[n][S]$ and $ED[n][S][i]$ where $i \in [1, n]$
$desiredSubsets = [\ ]$ ▷ $desiredSubsets$ are all the subsets of X_n with sum S.
$fullTable = GetED(n, S, ED[n][S], desiredSubsets)$ ▷ the count of every element in subsets of X_n with sum S called from function $GetED$ described in Algorithm 8
$p = SD[n][S]$: number of subsets of X_n with sum S

function GenerateSubsets
  if $S <= 0$ or $|elements| == 0$ **then**
    $Return\ desiredSubsets$
  end if
  $countSubsets = SD[n]$
  while $countSubsets > 0$ **do**
    $minVal = min(ED[n][S][e] > 0)$
    $minKey = max(e\ \forall e \in [1, n]\ \&\ ED[n][S][e] == minVal)$
    $elements.remove(minKey)$
    $elementsCovered.add(minKey)$
    $desiredSubsets = GeneratingSubsets(n, S - minKey, countSubsets, elements, elementsCovered, fullTable)$
    if $desiredSubsets[S - minKey]$ is empty **then**
      $countSubsets - -$
      $ED[n][S][minKey] - -$
      $desiredSubsets[S] = [[minKey]]$
      $print(desiredSubsets[S])$
      $elementsCovered.remove(minKey)$
    else
      for $A \in desiredSubsets[S - minKey]$ **do**
        $ED[n][S][minKey] - -$
        if $(minKey \notin A)\ \&\ (minKey + sum(A) == S)$ **then**
          if $A.append(minKey)\ is\ unique$ **then**
            $countSubsets - -$
            $print(A)$
            $desiredSubsets[S].append(A)$
            $In\ elementsCovered\ add\ elements\ of\ A$
          end if
        end if
      end for
    end if
  end while
  for $e \in ED[n][S][e] <= 0$ **do**
    $elementsCovered.add(e)$
  end for
  for $A \in desiredSubsets\ \&\ e \in A$ **do**
    $ED[n][S][e] + +$
  end for
  $Return\ desiredSubsets$
end function

Sum	*Length*	*Subsets*	*MaximalSubset*	*MinimalSubset*
0	0	ϕ	ϕ	ϕ
1	1	$\{\{1\}\}$	$\{1\}$	$\{1\}$
2	1	$\{\{2\}\}$	$\{2\}$	$\{2\}$
3	1	$\{\{3\}\}$	$\{3\}$	$\{3\}$
	2	$\{\{1,2\}\}$	$\{1,2\}$	$\{1,2\}$
4	1	$\{\{4\}\}$	$\{4\}$	$\{4\}$
	2	$\{\{1,3\}\}$	$\{1,3\}$	$\{1,3\}$
5	1	$\{\{5\}\}$	$\{5\}$	$\{5\}$
	2	$\{\{2,3\},\{1,4\}\}$	$\{2,3\}$	$\{1,4\}$
6	2	$\{\{2,4\},\{1,5\}\}$	$\{2,4\}$	$\{1,5\}$
	3	$\{\{1,2,3\}\}$	$\{1,2,3\}$	$\{1,2,3\}$
7	2	$\{\{3,4\},\{2,5\}\}$	$\{3,4\}$	$\{2,5\}$
	3	$\{\{1,2,4\}\}$	$\{1,2,4\}$	$\{1,2,4\}$
8	2	$\{\{3,5\}\}$	$\{3,5\}$	$\{3,5\}$
	3	$\{\{1,3,4\},\{1,2,5\}\}$	$\{1,3,4\}$	$\{1,2,5\}$
9	2	$\{\{4,5\}\}$	$\{4,5\}$	$\{4,5\}$
	3	$\{\{2,3,4\},\{1,3,5\}\}$	$\{2,3,4\}$	$\{1,3,5\}$
10	3	$\{\{2,3,5\},\{1,4,5\}\}$	$\{2,3,5\}$	$\{1,4,5\}$
	4	$\{\{1,2,3,4\}\}$	$\{1,2,3,4\}$	$\{1,2,3,4\}$
11	3	$\{\{2,4,5\}\}$	$\{2,4,5\}$	$\{2,4,5\}$
	4	$\{\{1,2,3,5\}\}$	$\{1,2,3,5\}$	$\{1,2,3,5\}$
12	3	$\{\{3,4,5\}\}$	$\{3,4,5\}$	$\{3,4,5\}$
	4	$\{\{1,2,4,5\}\}$	$\{1,2,4,5\}$	$\{1,2,4,5\}$
13	4	$\{\{1,3,4,5\}\}$	$\{1,3,4,5\}$	$\{1,3,4,5\}$
14	4	$\{\{2,3,4,5\}\}$	$\{2,3,4,5\}$	$\{2,3,4,5\}$
15	5	$\{\{1,2,3,4,5\}\}$	$\{1,2,3,4,5\}$	$\{1,2,3,4,5\}$

Table 8. Maximal and minimal subsets for every sum and length pair of X_5.

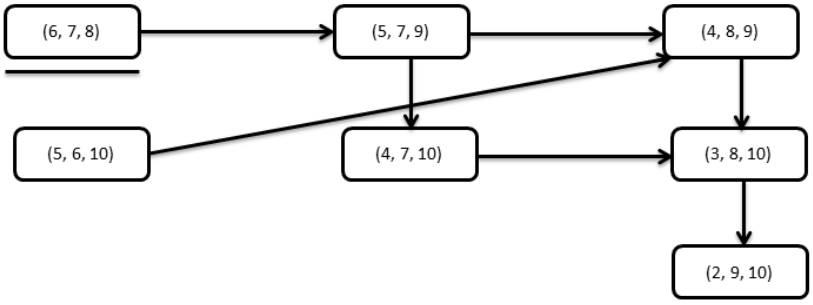


Fig. 1. Local search for $n = 10$, $sum = 21$ and $length = 3$ with maximal subset as the starting point.

second element of subset $\{6,7,8\}$, as on incrementing 7 by 1, we get 8 which creates duplication. In this case, 7 is a non-permissible element.

(b) Next, we generate subsets :$\{\{4,8,9\},\{5,6,10\},\{4,7,10\}\}$ from subset $\{5,7,9\}$.

(c) By following the same procedure, we generate all desired subsets of X_{10} with sum 21 and length 3 from a single maximal set $A_{maximal}$.

(2) Figure 2 presents the local search example for $n = 10$, $sum = 21$ and $length = 3$ where the starting subset is the minimal subset of respective length.

(a) We start with subset $\{2,9,10\}$. By incrementing the first permissible element 2 by 1 and decrementing the second permissible element 9 by 1, we generate the second subset $\{3,8,10\}$. We can not decrement the third element of subset $\{2,9,10\}$, as on decreasing 10 by 1, we get 9 which leads to duplication. In this case, 10 is a non-permissible element.

(b) Next, we generate subsets :$\{\{4,7,10\},\{4,8,9\}\}$ from subset $\{3,8,10\}$.

(c) By following the same procedure, we generate all desired subsets of X_{10} with sum 21 and length 3 from a single minimal set, $A_{minimal}$.

(3) While generating a subset using Local Search Algorithm, we ensure that the sum of subset is equal to the desired target sum S, the subset does not contain duplicates and there is uniqueness among the subsets. Uniqueness among and within these subsets is ensured by using lookup technique introduced in Section A. This establish the correctness of the Local Search Algorithms using Maximal and Minimal Subsets.

(4) Since we know the count of all subsets of X_n with $Sum = S$ and $Length = l$, we generate all the subsets and this approach is concluded only when all desired subset results are achieved. This establish the completeness of the Local Search Algorithms using Maximal and Minimal Subsets.

Local Search using Maximal Subset: Algorithm 11 presents a procedure to generate all subsets of X_n with particular sum S and length l where the seed subset is the maximal subset, $A_{maximal}$. We begin from the left most element, decrement the first permissible element followed by increment of next permissible element. Each increment or decrement consists of one unit. In Algorithm 11, we use a queue data structure to store all the resulting subsets, including maximal subset. We can iterate all the subsets in FCFS manner via these method. We check the uniqueness among the subsets by using the concept of lookup table as defined in Section A. A subset is pushed in the queue only if its unique. This algorithm is terminated when all the subsets are generated.

Local Search using Minimal Subset: Algorithm 12 represents a procedure to generate all subsets of X_n with particular sum S and length l where the seed subset is the minimal subset, $A_{minimal}$. We begin from left most element, increment the first permissible element followed by decrement of next permissible element. Every

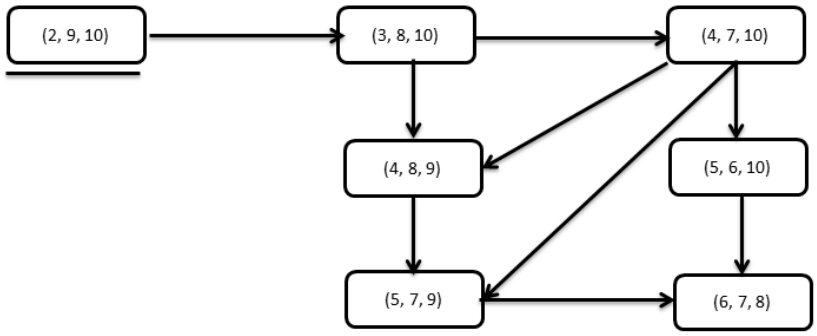


Fig. 2. Local search for $n = 10$, $sum = 21$ and $length = 3$ with minimal subset as the starting point.

increment or decrement consists of one unit. Algorithm 12 uses the same queue data structure and checks the uniqueness among the subsets by using the concept of lookup table as Algorithm 11. This algorithm is terminated when all subsets are generated.

Complexities: Time complexity of these algorithms is complexity of while loop × complexity of for loop, i.e., $maximum\ no.\ of\ subsets\ *\ length\ of\ each\ subset$. The complexity of the length of each subset variable is $O(n)$ but the time complexity of *maximum no. of subsets* variable is exponential. This makes the algorithm exhaustive. Time complexity is $O(2^n \cdot n^{\frac{-3}{2}} \cdot n) = O(2^n \cdot n^{\frac{-1}{2}}) = O(\frac{2^n}{\sqrt{n}})$. The space complexity for these algorithms is equal to the size of storage queue i.e. $maximum\ no.\ of\ subsets * length\ of\ each\ subset$. The time complexity is similar. The complexity of the $Length\ of\ each\ subset$ variable is $O(n)$ but the space complexity of *maximum no. of subsets* variable is exponential, $O(\frac{2^n}{\sqrt{n}})$.

6 EXPERIMENTAL RESULTS

This section presents the experiments that we have conducted to validate the efficiency and effectiveness of all the proposed algorithms.

6.1 Summary of Alternate Enumeration Techniques

Following table summarizes all the alternate enumeration techniques to solve SSP.

Problem Statement: Find all subsets of $\mathcal{P}(X_n)$ which sum up to S, where X_n is the set of first n natural numbers, $X_n = \{1, 2 \ldots n\}$			
Algorithm	**Core Idea**	**Time Complexity**	**Space Complexity**
Backtracking Algorithm (Naive) (section-5.1)	It is an improved and systematic brute force approach for generating various subsets with $Sum = S$. We iterate through all 2^n solutions in an orderly fashion.	$O(n \times 2^n)$	$O(n)$
Subset Generator using Sum Distribution (SDG) (section-5.2)	This algorithm is a recursive generator based on the concept of Sum Distribution and uses subsets of $X_{(n-1)}$ to produce results for X_n. Subsets of X_n with $Sum = S$ are generated by subsets of $X_(n-1)$ with $Sum = (S - n)$.	$O(2^n * n^{\frac{3}{2}})$	$O(2^n * n^{\frac{3}{2}})$

Subset Generator using Length-Sum Distribution (LDG) (section-5.3)	This algorithm is a recursive generator based on the concept of Length-Sum Distribution and uses subsets of $X_{(n-1)}$ to produce results for X_n. Subsets of X_n with $(Sum = S, Length = l)$ are generated by subsets of $X_{(n-1)}$ with $(Sum = S-n, Length = l-1)$.	$O(2^n * n^{\frac{5}{2}})$	$O(2^n * n^{\frac{5}{2}})$
Basic Bucket Algorithm (Basic BA) (section-5.4)	The basic idea behind this enumeration technique is to use the various distribution values. We consider $SD[n][S]$ number of empty buckets, storage data structures, and iterate through all elements in descending order. Each bucket represents a subset. During each iteration an element is assigned to one of the buckets. This method is about adding the correct element to the corresponding subset. This is an iterative algorithm.	$O(2^{2n} \cdot n^{-3})$	$O(2^n)$
Maximum Frequency Driven Bucket Algorithm (Max FD) (section-5.5)	Information used by this recursive algorithm is same as the basic bucket algorithm. Instead of choosing elements in descending order, we select largest element with maximum frequency to generate all $SD[n][S]$ number of subsets of X_n with $Sum = S$.	$O(2^{2n} \cdot n^{-3})$	$O(2^n)$
Minimum Frequency Driven Bucket Algorithm (Min FD) (section-5.5)	This algorithm is contrary to maximum FD bucket algorithm. Information used by this is also similar to the basic bucket algorithm. We select largest element with minimum frequency to generate all $SD[n][S]$ number of subsets of X_n with $Sum = S$.	$O(2^{2n} \cdot n^{-3})$	$O(2^n)$
Local Search using Maximal Subset (LS MaxS) (section-5.7)	This heuristic algorithm finds all desired subsets by choosing the maximal subset as the seed. Maximal subset has the largest possible element at every position for a given sum(S) and length(l). Therefore, we begin from left most element, decrement the first permissible element followed by increment of next permissible element. Every increment or decrement consists of one unit.	$O(\frac{2^n}{\sqrt{n}})$	$O(\frac{2^n}{\sqrt{n}})$
Local Search using Minimal Subset (LS MinS) (section-5.7)	This heuristic algorithm also finds all desired subsets by choosing the minimal subset as the seed (starting point). Minimal subset has smallest possible element at every position for a given sum(S) and length(l). Therefore, we begin from left most element, increment the first permissible element followed by decremental of next permissible element. Every increment or decrement consists of one unit.	$O(\frac{2^n}{\sqrt{n}})$	$O(\frac{2^n}{\sqrt{n}})$

Table 9. Summary of the core concepts and ideas of all the alternate enumeration techniques to solve subset sum problem. First column introduces every algorithm, second column presents the core idea behind the algorithm and the last two columns state their time and space complexities.

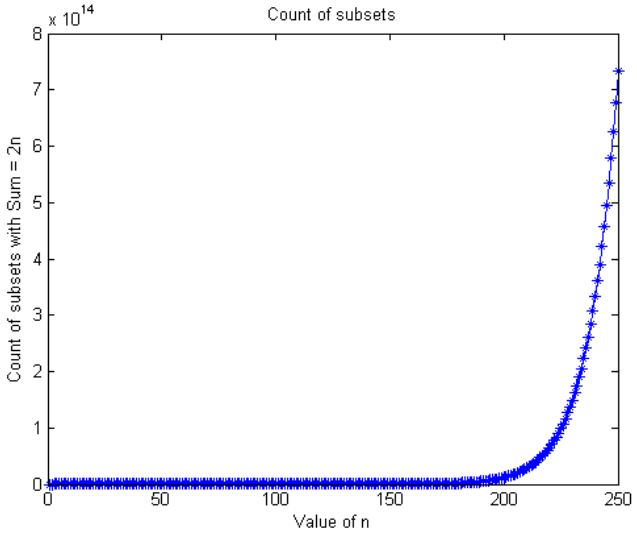
Algorithm 11 LS MaxS: Local Search for Maximal Subset

```
1: function localSearch(n, s, l)
2:     maximalSet =maximalSubset(n, s, l)
3:     queue.push(maximalSet)
4:     allSubsetsGenerated = LD[n][s][l]
5:     while allSubsetsGenerated > 0 do
6:         reqSet = queue.pop()
7:         for i = 1; i ≤ len − 1; i + + do
8:             if reqSet[i] − 1 > reqSet[i − 1] then
9:                 reqSet[i]− = 1                    ▷ First decrementing the element by 1
10:                decrement = True
11:            end if
12:            for j = i + 1; j ≤ l; j + + do
13:                if reqSet[j] + 1 < reqSet[j + 1] then
14:                    reqSet[j]+ = 1
15:                    increment = True
16:                end if
17:                if (reqSet is unique) and (decrement) and (increment) then
18:                    print reqSet
19:                    queue.push(reqSet)
20:                    allSubsetsGenerated − −
21:                end if
22:            end for
23:        end for
24:    end while
25: end function
```

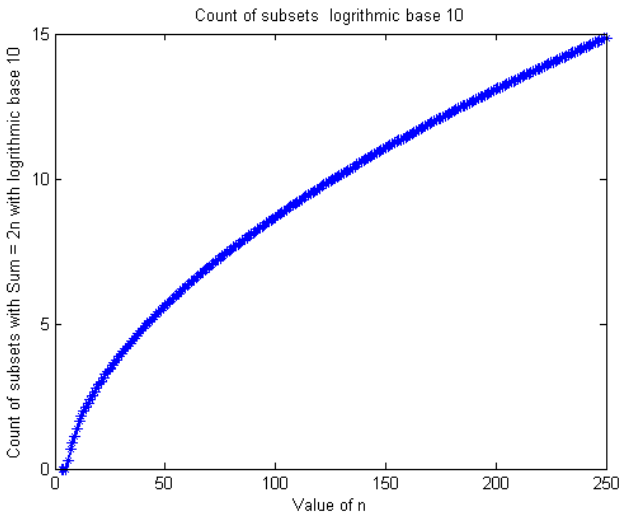
6.2 Experimental Setup

We have carried out various sets of experiments on an i7-2600 machine with 64GB of RAM to compare and analyze the performance of our algorithms under various considerations. We define the experimental setup and measuring parameters before comparing the performances.

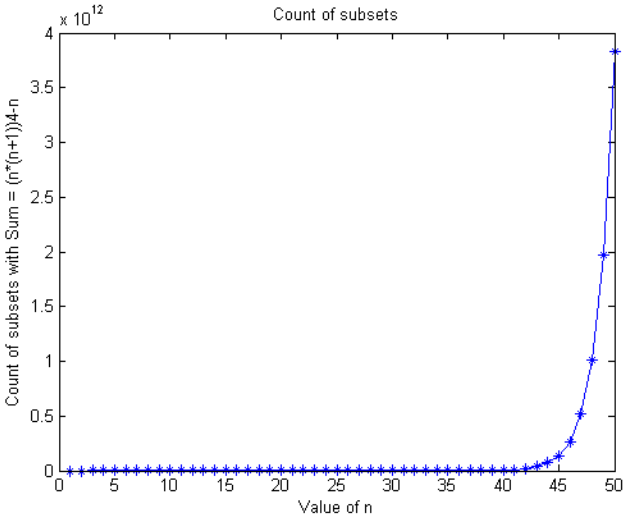
Due to symmetric property of SSP, we choose random sum values in lower part of the sum range i.e. $S \leq midSum(n)$. In Figure 3, we show different plots for number of subsets of X_n for various sums. These figures help us estimate the problem space for generating results of alternate enumeration techniques. We select the value of S as $2n$ to show the behavior of number of subsets of X_n with sum S when S has the complexity $O(n)$. Similarly, we choose the value of S as $midSum(n) - n$ because the number of subsets of X_n with this sum are in order of $O(midSum(n))$. This upper bound of the Sum Distribution $SD[n][midSum(n)] = S(n) \approx \sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{\frac{-3}{2}}$ is explained in Appendix B. Table 10 presents the count of number of subsets X_n with $S = 2n$ and $S = (\frac{n(n+1)}{4} - n)$. This table gives an estimate of the values plotted in Figure 3. Figure 3(a, c) plot the number of subsets of X_n with $(n \in [1,250], S = 2n)$ and $(n \in [1,50], S = (\frac{n(n+1)}{4} - n))$ respectively. Figure 3(b, d) plot the log to the base 10 of number of subsets of X_n with $(n \in [1,250], S = 2n)$ and $(n \in [1,50], S = (\frac{n(n+1)}{4} - n))$ respectively. Since the values of number of subsets for a particular S increases exponentially with n, we have plotted Figure 3(b, d) by using the logarithmic function. This helps in approximating the size of the problem space.



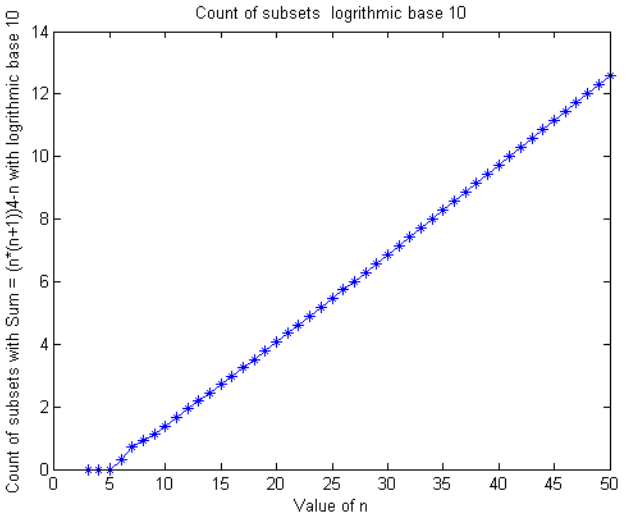
(a) Plot of number of subset of X_n at $S = 2n$ vs n



(b) Plot of log_{10}(number of subset of X_n) at $2n$ vs n



(c) Plot of number of subset of X_n at $S = (\frac{n(n+1)}{4} - n)$ vs n



(d) Plot of log_{10}(number of subset of X_n) at $S = (\frac{n(n+1)}{4} - n)$ vs n

Fig. 3. Plot of number of subsets of X_n against sums in smaller and larger ranges. For smaller range we select $S = 2n$ and plot graph for n varying from 1 to 250. (a) Figure represents graph for number of subsets of X_n with $S = 2n$ where $n \in [1,250]$. (b) Figure represents graph for number of subsets of X_n with $S = 2n$ with logarithmic base 10 where $n \in [1,250]$. For larger range we select $S = (\frac{n(n+1)}{4} - n)$ and plot graph for n varying from 1 to 50. (c) Figure represents graph for number of subsets of X_n with $S = (\frac{n(n+1)}{4} - n)$ where $n \in [1,50]$. (d) Figure represents graph for number of subsets of X_n with $S = (\frac{n(n+1)}{4} - n)$ with logarithmic base 10 where $n \in [1,50]$.

Algorithm 12 LS MinS: Local Search for Minimal Subset

```
1:  function localSearch(n, s, l)
2:      minimalSet =minimalSubset(n, s, l)
3:      queue.push(minimalSet)
4:      allSubsetsGenerated = LD[n][l][s]
5:      while allSubsetsGenerated > 0 do
6:          reqSet = queue.pop()
7:          for i = 1; i ≤ len − 1; i + + do
8:              if reqSet[i] + 1 < reqSet[i + 1] then
9:                  reqSet[i]+ = 1                    ▷ First incrementing the element by 1
10:                 increment = True
11:             end if
12:             for j = i + 1; j ≤ l; j + + do
13:                 if reqSet[j] − 1 > reqSet[j − 1] then
14:                     reqSet[j]− = 1
15:                     deccrement = True
16:                 end if
17:                 if (reqSet is unique) and (decrement) and (increment) then
18:                     print reqSet
19:                     queue.push(reqSet)
20:                     allSubsetsGenerated − −
21:                 end if
22:             end for
23:         end for
24:     end while
25: end function
```

6.3 Excess Subset Generation Analysis

Given X_n and a sum S, we know how many subsets of X_n have sum S. This value is $SD[n][S]$. For generating $SD[n][S]$ subsets by every algorithm, we may explore few extra subsets of X_n whose sum is not equal to S.

In naive backtracking method, at every step of subset generation we either include or exclude an element. This creates a recursive tree in which a branch is terminated when the current sum exceeds the target. This way we explore more subsets than desired number of subsets. Similarly, in rest of the alternate enumeration techniques in order to generate all subsets of X_n with sum S, we explore more subsets than desired number of subsets. In this analysis we measure this extra exploration. In Table 11 we present the ratios of subsets explored to total number of subsets of X_n with sum S i.e. $SD[n][S]$. The first three columns of this table states (n, S) pair and the value of $SD[n][S]$ for all these pairs. The remaining eight columns denote the ratio of explored subsets to the number of subsets in the final solution for all eight alternate enumeration techniques. With every ratio we also represent the time taken by an algorithm to generate the solution set. For every value of n and S, we bold the least ratio and least time taken by an algorithm.

Following observations can be made based on the data presented in Table 11:

(1) For a given value of n and S, desired ratio is a fraction of the number of subsets to be generated to the total number of subsets of X_n with Sum S i.e. $SD[n][S]$. For example, $n = 12$ and $S = 24$, the number of subsets of X_{12} with $Sum = 24$ are 67. Therefore, the value of $SD[12][24] = 67$. Therefore, the desired ratio for these values is: $\frac{67}{67} = 1$.

n	Count of Subsets of X_n with $S = 2n$	n	Count of Subsets of X_n with $S = (\frac{n(n+1)}{4} - n)$
6	2	6	2
7	5	7	5
8	8	8	8
9	13	9	13
10	134	10	24
50	416868	15	521
100	482240364	20	11812
150	114613846376	30	7206286
200	11954655830925	40	5076120114
250	732839540340934	50	3831141038816

Table 10. Count of number of subsets X_n with $S = 2n$ and $S = (\frac{n(n+1)}{4} - n)$ respectively.

(2) Every column corresponding to a given algorithm presents the ratio of number of subsets explored to generate the desired subsets to the total number of subsets of X_n with sum S. For example for naive algorithm, given $n = 12$ and $S = 24$, the number of subsets explored for generating all subsets of X_{12} with $Sum = 24$ are 737. Therefore, the ratio for these values is: $\frac{737}{67} = 11$.
(3) The ratio for all algorithms should be greater than the desired ratio. If not, then it implies that complete result has not been generated. In this table for a given n and S, the ratio for all algorithms is greater than the desired ratio. This observation and the correctness of these algorithms ensure the completeness of the results.
(4) Naive algorithm explores most number of subsets in order to generate the desired subsets. Naive is the worst performing enumeration technique compares to all our proposed algorithms. This shows that all our alternate enumeration techniques perform better than the benchmark algorithm.
(5) Ratios of LS MaxS and LS MinS are closer to the desired ratio for a given n and S.
    (a) Since LS MaxS and LSMinS are heuristic algorithms, they explore lesser number of subsets as compared to naive algorithm.
    (b) For example, given $n = 12$ and $S = 24$, the number of subsets explored for LS MaxS and LS MinS are 93 and 103 respectively creating a ratio of $\frac{93}{67} = 1.3881$ and $\frac{103}{67} = 1.5373$.
    (c) The drawback for these algorithm is that they do not generate results for higher values of n and S within short amount of time. This is explained more in Section 6.4.
(6) After Local Search algorithms, LDG and SDG are next in good performance ranking. Ratio for LDG is smaller and closer to desired ratio than SDG.
    (a) Since LDG is a simple dynamic algorithm which generate the subsets based on their sum and length, it goes to one more level of categorization among subsets and minimizes the excess exploration of undesired subsets.

(b) Given $n = 12$ and $S = 24$, the number of subsets explored by LDG are 150 and ratio is $\frac{150}{67} = 2.2388$.
(c) LDG has precedence over others as it can enumerate all subsets of X_n for a considerable values of n within short amount of time. The numbers are shown in Table 18 of Section 6.4.
(d) SDG explores more subsets than LDG but it performs better than naive. While naive implementation involves a recursive tree based on the inclusion and exclusion of an element at every step, SDG builds the subset by using the exact formula defined in Section 4.
(e) Given $n = 12$ and $S = 24$, the number of subsets explored by SDG are 214 and ratio is $\frac{214}{67} = 3.1940$.

(7) Performance of Max FD and Min FD is similar to SDG. For $n = 12$ and $S = 24$, MaxFD explores 166 subsets and Min FD explores 241 subsets. For other pairs of n and S these values are very close.
(8) Basic Bucket algorithm (Basic BA) also performs better than naive but can not compute all subsets for a considerable value of n and S within short amount of time.

We go beyond state of art to find exact solutions for this work. As shown in Table 11, we can summarize that these ratios are close to optimal numbers, in terms of the explored search space. Subsets are generated within optimal time by exploring concise and precise solution space. During generation of 1764 resulting subsets of $n = 17$ and $S = 59$, naive algorithm explores $\sim 47,800$ subsets while LS MaxS explores 1795 subsets.

n	S	$\|Subsets\|$	Naive	SDG	LDG	Basic BA	Max FD	Min FD	LS MaxS	LS MinS
12	24	67	11 (0.00247)	3.1940 (0.009596)	2.2388 **(0.00103)**	5.0896 (1.618)	2.4776 (0.195)	3.5970 (1.822)	**1.3881** (0.019)	1.5373 (0.016)
12	27	84	20.5952 (0.00178)	2.7857 (0.012512)	2.0952 **(0.00116)**	5.1548 (2.394)	2.6190 (0.405)	4.2262 (2.958)	1.3690 (0.02)	**1.2976** (0.024)
15	30	186	21.4194 (0.00795)	6.2097 (0.014521)	1.6882 **(0.00499)**	4.7742 (21.208)	2.3871 (1.381)	4 (7.641)	**1.1882** (0.11)	1.2903 (0.133)
15	45	521	23.3282 (0.01184)	2.8177 (0.056801)	1.3013 **(0.00472)**	- -	2.8503 (14.031)	1.3129 (353.746)	1.0211 (0.798)	**1.0058** (0.955)
16	32	253	60.6087 (0.01118)	8.2806 (0.017711)	1.6719 **(0.00615)**	5.4664 (77.48)	2.3478 (2.341)	3.7470 (11.761)	**1.1621** (0.211)	1.2332 (0.255)
17	59	1764	27.0947 (0.03996)	2.9127 (0.233748)	1.3622 **(0.01556)**	- -	2.9892 (153.31)	- -	**1.0176** (10.144)	1.1037 (12.058)
20	40	860	73.6390 (0.03277)	30.0447 (0.055301)	1.8189 **(0.04656)**	- -	2.2667 (21.923)	- -	**1.0707** (2.81)	1.1191 (3.438)
20	85	11812	28.4236 (0.30453)	2.9261 (2.08991)	1.6258 **(0.1357)**	- -	- (6981.574)	- -	1.2332 (572.839)	**1.2281** (664.875)
22	104	41552	34.7394 (1.21103)	2.9706 (8.53939)	**1.8080** **(0.52493)**	- -	- -	- -	- -	- -

Table 11. The ratios of subsets explored to total number of subsets of X_n with sum S i.e. $SD[n][S]$ is presented in this table. The first three columns of this table states (n, S) pair and the value of $SD[n][S]$ for all these pairs. The remaining eight columns denote subsets explored ratio for all eight alternate enumeration techniques. With every ratio we also represent the time taken by the algorithm to generate the solution set. The least ratio and least time taken for every n and S are presented in bold.

Experiments / Comparative Analysis	Description and Examples	Algorithms	Tables or Figures

CA-SSR $[1, 2n]$	For this experiment we randomly choose sum S_1 from a smaller range and calculate the time taken to generate subsets of X_n with $Sum = S_1$. For every values of n, this smaller range varies from 1 to $2n$ i.e. $\forall n,\ S_1 \in [1, 2n]$.	Basic BA, Max FD, Min FD, LS MaxS, LS MinS	Table 13: Time taken (in seconds) by Basic BA, Max FD and Min FD in CA-SSR. Table 15: Time taken (in seconds) by LS MaxS and LS MinS in CA-SSR.
CA-LSR $[midSum(n) - n, midSum(n)]$	For this experiment we randomly choose sum S_2 from a larger range and calculate the time taken by all the algorithms to generate subsets of X_n with $Sum = S_2$. For every values of n, this larger range varies from $midSum(n) - n$ to $midSum(n)$ i.e. $\forall n\ S_2 \in [midSum(n) - n, midSum(n)]$.	Basic BA, Max FD, Min FD, LS MaxS, LS MinS	Table 16: Time taken (in seconds) by Basic BA, Max FD and Min FD in CA-LSR. Table 14: Time taken (in seconds) by LS MaxS and LS MinS in CA-LSR.
CA-FSV	In this experiment instead of choosing random vales of S for every algorithm against every n, we fix few pairs of (n, S_1) and (n, S_2) for all the algorithms where $S_1 = 2 * n$ and $S_2 = midSum(n) - n$	Basic BA, Max FD, Min FD, LS MaxS, LS MinS, LDG, SDG	Table 17: presents the time taken by Max FD, Min FD, Basic BA, LS MaxS, LS MinS, LDG and SDG algorithms where $S_1 = 2 * n$ and $S_2 = midSum(n) - n$
CA-SLN	For this experiment instead of fixing the value of sum S, we vary S from 0 to $maxSum(n) = \frac{n(n+1)}{2}$. This experiment helps us analyze the performance of SDG and LDG algorithms against Naive (backtracking) algorithm. In this experiment we enumerate all 2^n subsets of X_n	SDG, LDG, Naive	Table 18: presents the comparison of SDG and LDG with naive backtracking algorithm. This table presents the time taken(in sec) while enumerating all 2^n subsets of X_n for every value of sum S in range $[0, \frac{n(n+1)}{2}]$. This is the time taken by these algorithms to enumerate each and every subset. Figure 4: Plot of SDG, LDG and Naive algorithm while enumerating all 2^n subsets of X_n for every value of sum S in range $[0, \frac{n(n+1)}{2}]$.

Table 12. Summary of the experimental setup for Comparative Analysis of Enumeration Algorithms. First column states the name, second columns describes the experiment, third column lists the algorithms for which the experiment has been carried out and the fourth column presents the tables and figures stating the time taken by different algorithms under several conditions.

6.4 Comparative Analysis of Enumeration Algorithms

In this section, we present the time taken by enumeration techniques under different conditions. Experiments defined in this section are categorized based on the range of input sum value corresponding to the set of natural numbers X_n. We are given X_n and $Sum(A)$ belonging to the range $[0, maxSum(n)] = [0, \frac{n(n+1)}{2}]$ where $A \in \mathcal{P}(X_n)$.

Choosing different values of sum between 0 to $maxSum(n)$ is the core idea behind these experiments. Table 12 summarizes the explanation of all these experiments.

We have drawn tables and shown the times for demonstrative purposes. Following observations are made from running all the alternate enumeration techniques:

(1) From comparative analysis of algorithms in smaller range (CA-SSR) we can see that Basic BA, Max FD, Min FD, LS MaxS and LS MinS generate subsets till n equal to 22, 36, 36, 44 and 44 respectively and takes less than 35,000 seconds.
   - Since LS MaxS and LS MinS explores lesser number of extra subsets as shown in Section 6.3, it takes lesser amount of time than bucket algorithms - Basic BA, Max FD and Min FD.
   - Among these five algorithms, Basic BA explores maximum number of subsets. It takes most time for execution and can generate results till smaller values of n, where n is less than 20.

(2) Comparative analysis of algorithms in larger range (CA-LSR) follows similar pattern as CA-SSR. The value of sum selected in this range has higher value of $SD[n][S]$ which results in higher execution time. LS MaxS and LS MinS perform the best in this experiment.

(3) Comparative Analysis with Fixed Sum Values (CA-FSV) allows us to compare seven algorithms: Max FD, Min FD, Basic BA, LS MaxS, LS MinS, LDG and SDG for a fixed value of n and S.
   - From this comparative study, we can see that LDG and SDG outperforms all the other algorithms. They can be executed till $n = 36$ and has lowest execution time.
   - Even though SDG and LDG explores more number of subsets, additional information required by these algorithms is much lesser than the additional information required by Local Search and Bucket Algorithms.
   - SDG and LDG does not require the values of $SD[n][S]$ and $ED[n][S][e]$ at every step of execution. We do not need to maintain the current state of these algorithms. This reduces the execution time.

(4) From comparative analysis of SDG, LDG and Naive (CA-SLN), we compare LDG, SDG with naive to show that our alternate enumeration techniques performs better than the existing algorithms. Using naive algorithm, we are not able to generate all the subsets for n greater than or equal to 24. This limits the execution. On the other hand, LDG and SDG can easily be computed till $n = 34$ in less than 40 minutes.

These timings are implementation and machine dependent. All these algorithms are tested on same the machine in this experimental setup. The above results show that even though some algorithms explore fewer extra subsets but they take more time due to lack of efficient implementation, storage and memory constraint.

n	S	$\|Subsets\|$	Max FD	Min FD	Basic BA	LS MaxS	LS MinS	LDG	SDG
12	24	67	0.195	1.822	1.618	0.019	0.016	0.00103	0.009596
12	27	84	0.405	2.958	2.394	0.02	0.024	0.00116	0.012512
14	28	134	0.808	3.95	14.76	0.066	0.065	0.00289	0.013285
14	38	274	3.9	54.175	161.639	0.215	0.256	0.00315	0.03134
15	30	186	1.381	7.641	21.208	0.11	0.133	0.00499	0.014521
15	45	521	14.031	353.746	1388.5	0.798	0.955	0.00472	0.056801
16	32	253	2.341	11.761	77.48	0.211	0.255	0.00615	0.017711
16	52	965	45.83	1224.328	-	2.882	3.394	0.0103	0.11678
17	34	343	4.414	27.817	236.119	0.412	0.501	0.00821	0.024524
17	59	1764	153.31	-	-	10.144	12.058	0.01556	0.233748
18	36	461	7.913	52.816	649.679	0.791	0.96	0.02192	0.034023
18	67	3301	541.046	-	-	39.129	45.385	0.03646	0.473109
20	40	806	21.923	146.823	-	2.81	3.438	0.04656	0.055301

20	85	11812	6981.574	-	-	572.839	664.875	0.1357	2.08991
21	42	1055	38.779	268.505	-	5.177	6.298	0.0664	0.072871
21	94	21985	25300.63	-	-	2084.648	2421.476	0.22368	4.134605
22	44	1369	64.492	842.423	-	9.411	11.516	0.09218	0.095904
22	104	41552	-	-	-	-	-	0.52493	8.53939
25	50	2896	295.741	-	-	52.604	64.216	0.57455	0.211722
25	137	283837	-	-	-	-	-	2.35755	73.2227
27	54	4649	831.93	-	-	155.258	190.273	1.06806	0.352428
27	162	1038222	-	-	-	-	-	10.77463	345.7571
30	60	9141	-	-	-	733.963	897.121	1.921185	-
30	202	7206286	-	-	-	-	-	98.64595	-

Table 17. The values of CA-FSV. We fix few pairs of (n, S_1) and (n, S_2) for Max FD, Min FD, Basic BA, LS MaxS, LS MinS, LDG and SDG algorithms where $S_1 = 2 * n$ and $S_2 = midSum(n) - n$. This table shows time taken (in seconds) by all alternate techniques for calculating for these pairs.

n	$\|Subsets\|$	LDG	SDG	Naive
2	4	0.00030	0.00018	0.0008
3	8	0.00024	0.00013	0.0012
4	16	0.00032	0.00015	0.0014
5	32	0.00037	0.00017	0.0026
6	64	0.00053	0.00023	0.0054
7	128	0.00061	0.00027	0.0116
8	256	0.00099	0.00032	0.0218
9	512	0.00122	0.00039	0.0423
10	1024	0.00218	0.00052	0.0854
11	2048	0.00257	0.00072	0.2137
12	4096	0.00391	0.00103	0.4542
13	8192	0.00532	0.00174	0.8522
14	16384	0.00863	0.00272	1.5655
15	32768	0.01223	0.00483	3.5153
16	65536	0.02119	0.00927	6.1746
17	131072	0.03060	0.01531	12.9676
18	262144	0.05147	0.02669	24.3167
19	524288	0.07002	0.05230	44.9257
20	1048576	0.12088	0.10512	92.8140
21	2097152	0.21260	0.20231	170.9037
22	4194304	0.44724	0.39577	364.9816
23	8388608	0.77863	0.81253	689.0156
24	16777216	1.64562	1.60156	-
25	33554432	3.01883	3.13995	-
26	67108864	6.22996	6.21826	-
27	134217728	11.55410	12.60573	-

28	268435456	23.83728	25.29129	-
29	536870912	46.01338	49.41213	-
30	1073741824	97.38387	98.06444	-
31	2147483648	184.78691	202.59311	-
32	4294967296	375.63728	407.96308	-
33	8589934592	755.37561	844.82139	-
34	17179869184	2130.2298	2363.4442	-

Table 18. Comparison of SDG and LDG with naive backtracking algorithm. This table presents the time taken(in sec) while enumerating all 2^n subsets of X_n for every value of sum S in range $[0, \frac{n(n+1)}{2}]$. This is the time taken by these algorithms to enumerate each and every subset in CA-SLN.

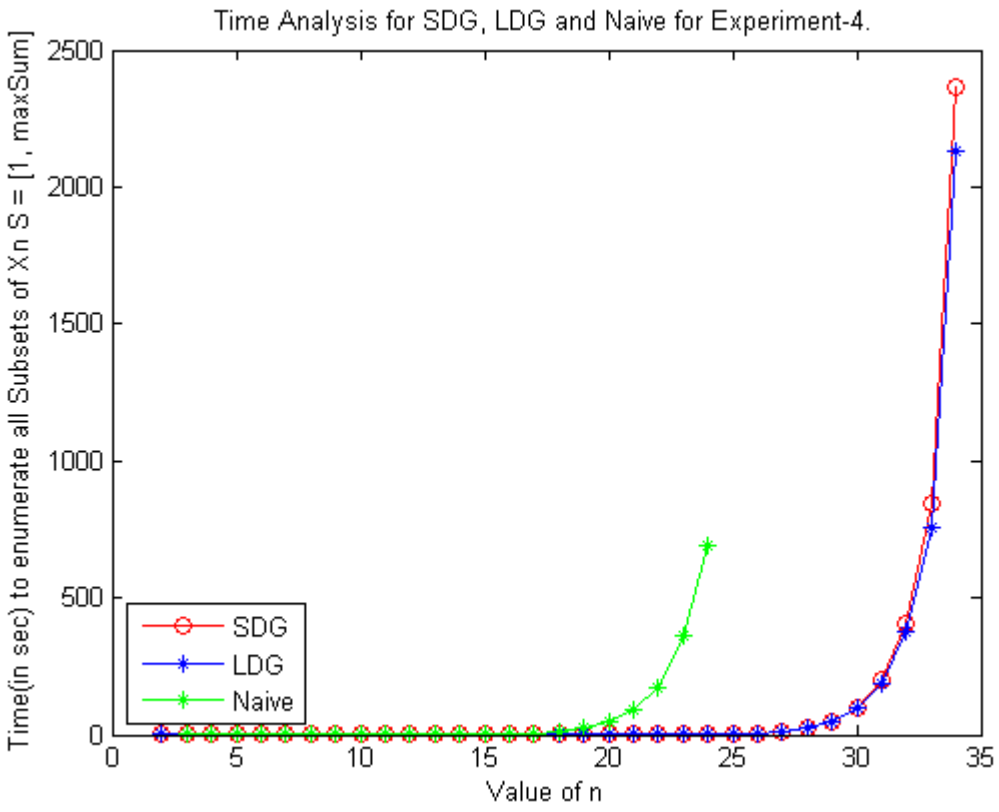


Fig. 4. Plot of SDG, LDG and Naive algorithm while enumerating all 2^n subsets of X_n for every value of sum S in range $[0, \frac{n(n+1)}{2}]$. This graph plots time taken by these algorithms to enumerate each and every subset in CA-SLN.

Time taken(in sec) by LS MaxS in CA-SSR					
n	S	Time(in sec)	n	S	Time(in sec)
3	1	0.00015	24	47	19.862
4	1	0.00012	25	36	2.0454
5	1	0.00015	26	10	0.0023
6	3	0.00024	27	17	0.0114
7	4	0.00023	28	5	0.002251
8	10	0.00098	29	8	0.002551
9	5	0.00039	30	8	0.00289
10	16	0.0037	31	53	168.747
11	20	0.0094	32	58	510.344
12	11	0.0018	33	31	1.00335
13	6	0.00070	34	56	411.957
14	1	0.00060	35	50	124.164
15	17	0.01081	36	47	67.4379
16	4	0.00086	37	62	1748.339
17	32	0.30375	38	74	18096.70
18	17	0.00682	39	56	588.9686
19	33	0.46984	40	58	951.2177
20	10	0.00144	41	32	1.890367
21	28	0.19325	42	55	561.6479
22	14	0.00403	43	46	76.02470
23	21	0.03176	44	44	48.61702

Time taken(in sec) by LS MinS in					
n	S	Time(in sec)	n	S	Time(in sec)
3	1	0.00020	24	47	21.2260
4	1	0.00019	25	19	0.01893
5	1	0.00020	26	52	77.8854
6	2	0.00022	27	22	0.05621
7	7	0.00073	28	10	0.00278
8	6	0.00049	29	47	41.2712
9	9	0.00092	30	58	434.903
10	4	0.00047	31	10	0.00344
11	22	0.01392	32	12	0.00514
12	9	0.00122	33	10	0.00396
13	10	0.00145	34	27	0.356276
14	3	0.00069	35	43	26.97922
15	15	0.00634	36	44	36.25960
16	26	0.09706	37	32	1.74147
17	20	0.02005	38	34	3.16681
18	11	0.00148	39	59	1143.04
19	28	0.15588	40	13	0.00915
20	10	0.00149	41	55	556.808
21	31	0.43241	42	26	0.35234
22	28	0.22006	43	14	0.01238
23	20	0.02313	44	40	18.9293

Table 13. Time taken (in seconds) by Local Search using Maximal Subset (LS MaxS) and Local Search using Minimal Subset (LS MinS) in CA-SSR where S_1 is randomly chosen and $\forall n,\ S_1 \in [1, 2n]$.

Time taken(in sec) by LS MaxS in CA-LSR					
n	S	Time(in sec)	n	S	Time(in sec)
3	1	0.00003	13	39	0.12096
4	2	0.00025	14	49	0.34326
5	5	0.00057	15	47	0.77065
6	6	0.00049	16	54	2.65762
7	8	0.00067	17	72	10.4998
8	18	0.001429	18	67	30.6323
9	20	0.003262	19	87	149.328
10	18	0.00491	20	97	649.048
11	23	0.01419	21	94	1635.28
12	30	0.04604	22	114	8633.37

Time taken(in sec) by LS MinS in CA-LSR					
n	S	Time(in sec)	n	S	Time(in sec)
3	1	0.00062	13	41	0.12125
4	2	0.00374	14	50	0.33774
5	6	0.00473	15	56	0.87691
6	7	0.01077	16	56	2.99347
7	11	0.00115	17	66	11.1003
8	13	0.0102	18	68	32.6975
9	14	0.00251	19	92	124.407
10	23	0.00898	20	100	639.238
11	29	0.02194	21	114	1881.25
12	31	0.05642	22	116	8897.909

Table 14. Time taken (in seconds) by Local Search using Maximal Subset (LS MaxS) and Local Search using Minimal Subset (LS MinS) in CA-LSR where S_2 is randomly chosen and $\forall n,\ S_2 \in [midSum(n) - n, midSum(n)]$.

7 CONCLUSION

Subset Sum Problem, also referred as SSP, is a well-known important problem in computing, cryptography and complexity theory. We extended the traditional SSP and designed alternate enumeration techniques. Instead of finding one subset with target sum, we find all possible solution of SSP. Therefore, for $X = \{5, 4, 9, 11\}$ and $S = 9$, the solution to our version of SSP is both $\{5, 4\}$ and $\{9\}$. We confined our problem domain by considering first n natural numbers as set X_n. In other words, we enumerate all $(2^n - 1)$ power set of a set.

n	S	Time taken(in sec) by Basic BA in CA-SSR
3	1	0.000551
4	1	0.0004.20
5	2	0.000138
6	3	0.000247
7	5	0.000405
8	2	0.000849
9	2	0.000885
10	16	0.05103
11	13	0.01842
12	13	0.02192
13	9	0.00265
14	16	0.08399
15	26	7.03249
16	31	60.6872
17	34	241.571
18	6	0.00106
19	19	0.75584
20	36	1261.39
21	15	0.09077
22	41	12918.6

n	S	Time taken(in sec) by Max FD in CA-SSR
3	1	0.00076
4	1	0.000564
5	1	0.000569
6	3	0.001332
7	6	0.003052
8	10	0.000837
9	12	0.00139
10	7	0.00384
11	20	0.12669
12	24	0.19757
13	24	0.220844
14	5	0.0011010
15	2	0.0003778
16	9	0.0040118
17	11	0.007174
18	7	0.0024759
19	3	0.0008509
20	33	4.143428
21	11	0.007366
22	37	11.79708
23	29	1.868481
24	42	42.95121
25	22	0.274363
26	4	0.0009
27	5	0.001708
28	29	2.35588
29	27	1.51263
30	13	0.029837
31	15	0.068043
32	17	0.113950
33	57	1822.731
34	47	237.329
35	36	21.1548
36	71	32840.56

n	S	Time taken(in sec) by Min FD in CA-SSR
3	1	0.00094
4	1	0.00065
5	2	0.00072
6	4	0.00156
7	1	0.00073
8	1	0.00073
9	3	0.00016
10	3	0.00172
11	7	0.00503
12	24	0.29195
13	21	0.14098
14	10	0.00586
15	23	0.25498
16	22	0.21270
17	34	4.35665
18	15	0.04607
19	20	0.15957
20	15	0.03706
21	9	0.00526
22	6	0.00236
23	34	8.08626
24	42	52.7242
25	9	0.00573
26	29	2.61190
27	33	8.01348
28	13	0.03223
29	51	523.152
30	24	0.73111
31	33	9.51999
32	41	71.6738
33	43	117.805
34	23	0.71141
35	14	0.08665
36	70	33113.13

Table 15. Time taken (in seconds) by Basic Bucket Algorithm (Basic BA), Maximum Frequency Driven Bucket Algorithm (Max FD) and Minimum Frequency Driven Bucket Algorithm (Min FD) in CA-SSR where S_1 is randomly chosen and $\forall n$, $S_1 \in [1, 2n]$.

We have analyzed the distribution of $\mathcal{P}(X_n)$ over sum, length and count of individual elements. We introduced four types of distributions: Sum Distribution, Length Distribution, Length-Sum Distribution and Element Distribution. We extended the concept by explaining their formulae and algorithms, along with illustrations, which showed a definite pattern and relations among these subsets. These distributions are prepossessing procedures for alternate enumeration techniques for solving SSP.

We developed Backtracking Algorithm (Naive) algorithm. It is an improved and systematic brute force approach for generating various subsets with $Sum = S$. Instead of searching exhaustively elements are selected systematically. We iterate through all 2^n solutions in this an orderly fashion. The inputs for this algorithm are

n	S	Time taken(in sec) by Basic BA in CA-LSR
4	3	0.00079
5	6	0.00102
6	4	0.00044
7	12	0.00882
8	17	0.02263
9	18	0.11157
10	25	0.32170
11	31	1.59752
12	35	9.34307
13	40	39.5506
14	48	437.383
15	50	1846.33

n	S	Time taken(in sec) by Max FD in CA-LSR
3	3	0.00143
4	5	0.00153
5	7	0.00252
6	10	0.00522
7	14	0.01324
8	18	0.02944
9	22	0.07514
10	27	0.15790
11	33	0.41121
12	39	1.38157
13	45	4.16112
14	52	12.2192
15	60	39.9648
16	68	132.079
17	76	434.065
18	85	1426.70
19	95	4850.73
20	105	17189.86

n	S	Time taken(in sec) by Min FD in Exp-2
3	2	0.00081
4	4	0.00129
5	3	0.00117
6	6	0.00329
7	7	0.00451
8	10	0.01452
9	13	0.03446
10	21	0.20658
11	31	1.87448
12	37	12.5400
13	33	7.19542
14	46	166.192
15	57	793.294

Table 16. Time taken (in seconds) by Basic Bucket Algorithm (Basic BA), Maximum Frequency Driven Bucket Algorithm (Max FD) and Minimum Frequency Driven Bucket Algorithm (Min FD) in CA-LSR where S_2 is randomly chosen and $\forall n,\ S_2 \in [midSum(n) - n, midSum(n)]$.

the set of first n natural numbers X_n and $Sum = S$. Time and space complexities for this algorithm are $O(n \times 2^n)$ and $O(n)$ respectively.

We have proposed Subset Generator using Sum Distribution(SDG). This algorithm is a recursive generator based on the concept of Sum Distribution and uses subsets of $X_{(n-1)}$ to produce results for X_n. This algorithm uses the formula defined in Equation 1. This algorithm is executed using dynamic programming. Subsets of X_n with $Sum = S$ are generated by subsets of X_{n-1} with $Sum = S$ and $Sum = (S - n)$. Time and space complexities for this algorithm are $O(2^n * n^{\frac{3}{2}})$ and $O(2^n * n^{\frac{3}{2}})$ respectively.

We have proposed Subset Generator using Length-Sum Distribution (LDG). This algorithm is a recursive generator based on the concept of Length-Sum Distribution and uses subsets of $X_{(n-1)}$ to produce results for X_n. This algorithm uses the formula defined in Equation 2. This algorithm is executed using dynamic programming. Subsets of X_n with $(Sum = S, Length = l)$ are generated by subsets of X_{n-1} with $(Sum = S, Length = l)$ and $(Sum = S - n, Length = l - 1)$. Time and space complexities for this algorithm are $O(2^n * n^{\frac{5}{2}})$ and $O(2^n * n^{\frac{5}{2}})$ respectively.

We have also proposed Basic Bucket Algorithm (Basic BA). The basic idea behind this enumeration technique is to use the various distribution values. We consider $SD[n][S]$ number of empty buckets, storage data structures, and iterate through all elements in descending order. It uses the value of Element Distribution for generating all the desired subsets. During each iteration an element is assigned to one of the buckets. This method is about adding the correct element to the corresponding subset. This is a greedy algorithm. This method uses the concept of lookup table explained in Section A and ensures uniqueness among and within the subsets. Time and space complexities for this algorithm are $O(2^{2n} \cdot n^{-3})$ and $O(2^n)$ respectively.

Next, we have extended the concept of Basic Bucket Algorithm (Basic BA) to propose two new bucket algorithms: Maximum Frequency Driven Bucket Algorithm (Max FD) and Minimum Frequency Driven Bucket Algorithm

(Min FD). Information used by these recursive algorithms are same as the basic bucket algorithm. For Max FD, instead of choosing elements in descending order, we select maximum element with maximum frequency to generate all $SD[n][S]$ number of subsets of X_n with $Sum = S$. For Min FD we select maximum element with minimum frequency to generate all $SD[n][S]$ number of subsets of X_n with $Sum = S$. These methods use the concept of lookup table explained in Section A and ensure uniqueness among and within the subsets. Time and space complexities for this algorithm are $O(2^{2n} \cdot n^{-3})$ and $O(2^n)$ respectively.

We have proposed two more algorithms Local Search using Maximal Subset (LS MaxS) and Local Search using Minimal Subset (LS MinS). Maximal and Minimal Subsets are a new idea for categorizing subsets of a given class. First, we divide the power set of X_n, $\mathcal{P}(X_n)$, on the basis of their sum and then further partition these subsets according to their length. LS MaxS is a heuristic algorithm. It finds all the desired subsets by choosing the maximal subset as the seed. Maximal subset has largest possible element at every position for a given sum(S) and length(l). Therefore, we begin from left most element, decrement the first permissible element followed by increment of next permissible element. LS MinS is also a heuristic algorithm also finds all desired subsets by choosing the minimal subset as the seed. Minimal subset has the smallest possible element at every position for a given sum(S) and length(l). Therefore, we begin from left most element, increment the first permissible element followed by decremental of next permissible element. Every increment or decrement consists of one unit. Time and space complexities for this algorithm are $O(\frac{2^n}{\sqrt{n}})$ and $O(\frac{2^n}{\sqrt{n}})$ respectively.

Conjecture. *There are algorithms that can enumerate all solutions of Subset Sum Problem for set X_n and sum S where $0 \leq S \leq \frac{n(n+1)}{2}$ with $O(SD[n][S])$ complexity.*

An optimal algorithm should enumerate exactly $SD[n][S]$ subsets which are part of the solution.

This work can be extended in following ways:

(1) By amortizing and combining different set of sums as one input set. Instead of running one sum at a time, we can group the sum values for running alternate enumeration techniques. This will save the execution time by avoiding recalculations of subsets for smaller ranges.
(2) Additionally, we can reduce the execution time of alternate enumeration techniques. These techniques are implementation and machine dependent. These timings are also data structure dependent. As part of future work, we would like to explore more data structures and more powerful machines to reduce the running times furthermore.
(3) We have seen that the Local Search algorithm using Maximal or Minimal Subset comparatively explores less number of extra subsets and have better execution time than bucket algorithms. We can enhance this algorithm by using element distribution to limit the heuristic search, by finding different starting points and applying better distance formula for traversing through the solution space.

A LOOKUP TECHNIQUE

Mapping of each subset with a unique integer is the basic concept used to define a lookup table for power sets of X_n, where $X_n = \{1, 2 \ldots n\}$. Lookup table ensures uniqueness among the subsets and within elements for a subset. This table helps us to maintain the uniqueness at runtime of any algorithm. This technique is implemented with the help of bit vectors. Bit vector is a compact data structure which hashes each subset $A = \{A_1, A_2 \ldots A_l\}$ to the corresponding integer, denoted by *num*, $S_{num} = \sum_{i=1}^{l} 2^{A_i - 1}$. We consider a hash of size 2^n. This hash will maintain a one-to-one mapping between all the subsets of X_n and is denoted by $\mathcal{P}(X_n)$.

B UPPER BOUND ON SUM DISTRIBUTION

In this section, we use definitions and formulas presented in 3. By using the maximum limit on the number of subsets with a particular sum, we find an upper bound of our problem.

$SD[n]$, defined in Section 4.1 represents the count of all the subsets of X_n divided over sum S where $S \in [1, b]$ and $b = \frac{n(n+1)}{2}$ (Table 1). The maximum value of $SD[n]$ is found at $midSum(n) = \lfloor \frac{n(n+1)}{4} \rfloor$. Table 19 represents the value of $SD[n][midSum(n)]$ for first 15 natural numbers.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$sd[n][midSum(n)]$	1	1	2	2	3	5	8	14	23	40	70	124	221	397	722

Table 19. Values of $SD[n][midSum(n)]$ for first 15 natural numbers

For each n, value of $SD[n][midSum(n)]$ presented in table 19 is the coefficient of $x^{\frac{n(n+1)}{4}}$ in the expansion of $\{(1+x)(1+x^2)(1+x^3)\dots(1+x^n)\}$. This coefficient is denoted as $S(n)$ and $S(n) \approx \sqrt{\frac{6}{\pi}} \cdot 2^n \cdot n^{\frac{-3}{2}}$ [16]. Therefore, value of maximum number of subsets with sum as $midSum(n)$ has exponential bound, $O(2^n \cdot n^{\frac{-3}{2}})$. This result is vastly used throughout the paper in order to find complexities of enumeration techniques.

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