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by

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Report No: IIIT/TR/2019/-1

Centre for Robotics
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Hyderabad - 500 032, INDIA
December 2019

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Abstract

Available control methods for underactuated Euler-Lagrange (EL) systems rely on structure-specific constraints that may be appropriate for some systems, but restrictive for others. A generalized (structure-independent) control framework is to a large extent missing, especially in the presence of uncertainty. This paper introduces an adaptive-robust control framework for a quite general class of uncertain underactuated EL systems. Compared to existing literature, the important attributes of the proposed solution are: (i) avoiding structure-specific restrictions, namely, symmetry condition property of the mass matrix, and a priori bounds on non-actuated states or state derivatives; (ii) considering Coriolis, centripetal, friction and gravity terms to be unknown, while only requiring the knowledge of maximum perturbation around a nominal value of the mass matrix; (iii) handling state-dependent uncertainties irrespective of their linear or nonlinear in parameters structure. These features significantly widen the range of underactuated EL systems the proposed solution can handle in comparison to the available methods. Stability is studied analytically and the performance is verified in simulation using offshore boom crane dynamics.

Key words: Adaptive-robust control; Euler-Lagrange systems; Underactuated systems.

1 Introduction

Euler-Lagrange (EL) dynamics are used to describe a large range of real-world systems like robotic manipulators, mobile robots, underwater vehicles, aircrafts, satellites, among others [1–5]. *Underactuation* arises in these systems whenever the independent control inputs are less than their degrees of freedom. Underactuation can be: inherent in the system dynamics (e.g., cranes [6], aircrafts, helicopters [7]); desirable for cost reduction (e.g., satellites with fewer thrusters [8]); imposed for benchmarking (e.g., Acrobot [9], Pendubot [10]). Despite their merits in terms of costs and operational flexibility, underactuated EL, as compared to fully-actuated EL systems, come with greater challenges in terms of control.

Underactuated EL systems have received extensive attentions. Application-specific works [6, 10–16] have appeared along with theoretic works on special classes of underactuated dynamics [7–9, 17–21]. Works in the first category rely on structure-specific physical constraints that may be appropriate for some systems, but restrictive for others (e.g. prior assumption for a crane on bounded non-actuated state [6, 11, 13] is eventually restrictive for a satellite [8]). For this reason, finding generalized (structure-independent) control solutions for underactuated EL systems is a challenging open problem. Even most works in the second category ([7, 17–20]) rely on a structural *symmetry condition* of the mass matrix (i.e., the mass matrix being function of only the actuated states or only the non-actuated states), stemming from the pioneering work [7]. Such condition becomes difficult to be satisfied as the system complexity rises (e.g., cranes [6, 11, 13], underwater vehicles [22], satellites [8]).

On top of this, parametric uncertainties in underactuated EL system dynamics terms (mass, Coriolis, centripetal, friction and gravity terms), inevitable in practical scenarios, increase the control challenge. While [19] gives an overview of control approaches aimed at tackling uncertainty in a *robust* sense, such approaches either require precise knowledge of such terms [19–21], or perturbation bounds around nominal values [8, 16]. Such a priori system knowledge is often difficult to obtain. In light of this overview, a control design for uncertain underactuated EL systems removing several restrictions in terms of structural constraints and a priori knowledge of

⋆ This work was partly supported by the Fundamental Research Funds for the Central Universities Grant 4007019109, and by the special guiding funds for double first-class Grant 4007019201. Corresponding author: S. Baldi

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system dynamics, is relevant and missing. In pursuit of this objective, this paper proposes an *Adaptive-Robust Control* (ARC) framework for a class of uncertain underactuated EL systems with three main contributions:

- The proposed solution does not rely on symmetry condition of mass matrix and it does not assume non-actuated states to be bounded.
- Coriolis, centripetal, friction and gravity terms are considered completely unknown; only a perturbation bound around a nominal value of the mass matrix is required for control design.
- Compared to ARC advances [23–25] (dealing with fully-actuated dynamics), the proposed ARC tackles underactuated dynamics in both linear-in-parameters (LIP) and nonlinear-in-parameters (NLIP) form.

The class under consideration covers several underactuated EL systems addressed in literature, with the exception of non-holonomic EL systems (cf. Remark 3).

The following notations are used: $\lambda_{\min}(\bullet)$ and $||\bullet||$ represent minimum eigenvalue and Euclidean norm of $(\bullet)$ respectively; $\mathbf{I}_n$ denotes identity matrix with dimension $n \times n$; $\mathbf{D} > \mathbf{0}$ denotes a positive definite matrix $\mathbf{D}$.

2 System Dynamics and Problem Formulation

Consider the following class of underactuated Euler-Lagrange (EL) systems

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}}+\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}}+\mathbf{G}(\mathbf{q})+\mathbf{F}(\dot{\mathbf{q}})+\mathbf{d_s} = [\boldsymbol{\tau}^T \ \mathbf{0}^T]^T, \quad (1)$$

where $\mathbf{q} \in \mathbb{R}^n$ is the vector of generalized coordinates (positions), and $\dot{\mathbf{q}} \in \mathbb{R}^n$ is the vector of generalized velocities; $\mathbf{M}(\mathbf{q}) \in \mathbb{R}^{n\times n}$ is the mass/inertia matrix; $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) \in \mathbb{R}^{n\times n}$ denotes the Coriolis, centripetal terms; $\mathbf{G}(\mathbf{q}) \in \mathbb{R}^n$ denotes the gravity terms; $\mathbf{F}(\dot{\mathbf{q}}) \in \mathbb{R}^n$ denotes the vector of damping and friction forces; $\mathbf{d_s}(t) \in \mathbb{R}^n$ denotes bounded external disturbances and $\boldsymbol{\tau} \in \mathbb{R}^m$ is the generalized control input, where $(n-m) \le m < n$. For most EL systems of practical interest, a few properties can be claimed (cf. [1]) which are later exploited for control design as well as stability analysis:

Property 1: $\exists c_b, g_b, f_b, \bar{d} \in \mathbb{R}^+$ such that $||\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})|| \le c_b||\dot{\mathbf{q}}||$, $||\mathbf{G}(\mathbf{q})|| \le g_b$, $||\mathbf{F}(\dot{\mathbf{q}})|| \le f_b||\dot{\mathbf{q}}||$ and $||\mathbf{d_s}(t)|| \le \bar{d}$.
Property 2: $\mathbf{M}(\mathbf{q})$ is symmetric [1] and uniformly positive definite. This implies that $\exists \mu_1, \mu_2 \in \mathbb{R}^+$ such that

$$0 < \mu_1\mathbf{I}_n \le \mathbf{M}(\mathbf{q}) \le \mu_2\mathbf{I}_n. \quad (2)$$

Further, consider the decomposition of $\mathbf{M}$ as $\mathbf{M} = \hat{\mathbf{M}} + \Delta\mathbf{M}$, where $\hat{\mathbf{M}}$ and $\Delta\mathbf{M}$ represent the nominal and perturbation terms of the mass matrix, respectively. The EL system (1) is considered to be uncertain in the sense that precise parametric knowledge of the system dynamics terms $\mathbf{M}, \mathbf{C}, \mathbf{F}, \mathbf{G}$ and $\mathbf{d_s}$ is not available. The following challenge is imposed in the form of an assumption:

[1] The term "symmetric" is not to be confused with the "symmetry condition" proposed in [7, 17–20], cf. Remark 1.

Assumption 1 *Only a nominal* $\hat{\mathbf{M}}$ *and an upper bound for* $\Delta\mathbf{M}$ *are available, while the terms* $\mathbf{C}, \mathbf{F}, \mathbf{G}$, $\mathbf{d_s}$ *and their upper bounds* c_b, f_b, g_b *and* $\bar{d}$ *are unknown.*

Remark 1 *No assumption is made on how actuated and non-actuated states affect* $\mathbf{M}, \mathbf{C}, \mathbf{F}, \mathbf{G}$ *or* $\mathbf{d_s}$. *Therefore, restrictive assumptions such as symmetry condition of mass matrix, i.e.,* $\mathbf{M}(\mathbf{q}) = \mathbf{M}(\mathbf{q_a})$ *or* $\mathbf{M}(\mathbf{q}) = \mathbf{M}(\mathbf{q_u})$ *([7, 17–20]) or boundedness of state derivatives ([20]) or of non-actuated states ([6, 11, 13]) are completely avoided.*

For controller design, as well as for convenience of notation, let us rewrite system (1) by distinguishing between the actuated and non-actuated dynamics, as follows:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{N}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{d_s} = [\boldsymbol{\tau}^T \ \mathbf{0}^T]^T, \quad (3)$$

where $\mathbf{q} = [\mathbf{q_a}^T \ \mathbf{q_u}^T]^T$, being $\mathbf{q_a} \in \mathbb{R}^m$ the actuated states, $\mathbf{q_u} \in \mathbb{R}^{(n-m)}$ the non-actuated states and

$$\mathbf{M} \triangleq \begin{bmatrix} \mathbf{M_{aa}} & \mathbf{M_{au}} \\ \mathbf{M_{au}^T} & \mathbf{M_{uu}} \end{bmatrix}, \quad \begin{matrix} \mathbf{M_{aa}} \in \mathbb{R}^{m\times m}, \mathbf{M_{au}} \in \mathbb{R}^{m\times(n-m)} \\ \mathbf{M_{uu}} \in \mathbb{R}^{(n-m)\times(n-m)} \end{matrix},$$

$$\mathbf{N} \triangleq \mathbf{C}\dot{\mathbf{q}} + \mathbf{G} + \mathbf{F} = \begin{bmatrix} \mathbf{N_a^T} & \mathbf{N_u^T} \end{bmatrix}^T, \mathbf{N_a} \in \mathbb{R}^m, \mathbf{N_u} \in \mathbb{R}^{(n-m)},$$

$$\mathbf{d_s} \triangleq \begin{bmatrix} \mathbf{d_a^T} & \mathbf{d_u^T} \end{bmatrix}^T, \mathbf{d_a} \in \mathbb{R}^m, \mathbf{d_u} \in \mathbb{R}^{(n-m)}. \quad (4)$$

Dependency of the system dynamic terms on $(\mathbf{q},\dot{\mathbf{q}})$ has been and will be omitted whenever convenient. After suitable mathematical rearrangements, the system dynamics (3) can further be represented as [9, 21]

$$\ddot{\mathbf{q}}_\mathbf{u} = -\mathbf{M_{uu}^{-1}}\mathbf{M_{au}^T}\ddot{\mathbf{q}}_\mathbf{a} + \mathbf{h_u}, \quad (5a)$$

$$\ddot{\mathbf{q}}_\mathbf{a} = \mathbf{M_s^{-1}}\boldsymbol{\tau} + \mathbf{h_a}, \quad (5b)$$

where

$$\begin{aligned} \mathbf{h_u} &\triangleq \mathbf{M_{uu}^{-1}}(\mathbf{N_u} + \mathbf{d_u}), \\ \mathbf{h_a} &\triangleq \mathbf{M_s^{-1}}(\mathbf{N_a} + \mathbf{d_a} - \mathbf{M_{au}}\mathbf{M_{uu}^{-1}}(\mathbf{N_u} + \mathbf{d_u})), \\ \mathbf{M_s} &\triangleq \mathbf{M_{aa}} - \mathbf{M_{au}}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}^T}. \end{aligned}$$

As $\mathbf{M} > \mathbf{0}$ by *Property 2*, existence of $\mathbf{M_s^{-1}}, \mathbf{M_{aa}^{-1}}$ and $\mathbf{M_{uu}^{-1}}$ is always ensured [8, 9, 21].

Remark 2 *The dynamics under consideration are a class of EL dynamics, in view of having* $(n-m) \le m$. *Most EL systems of practical interest reported in literature belong to such class (e.g., Acrobot [9], Pendubot [10], aircrafts, helicopters [7], quadrotors [14], crane systems [11–13, 15, 19, 26], satellites [8], ships and underwater* *vehicles [22] etc.). Nevertheless, the class under consideration does not cover having dependent coordinates (nonholonomic EL systems [16]), or singular mass matrices arising from such dependent coordinates [27, 28].*

Remark 3 *In the absence of the unmatched disturbance $\mathbf{d_u}$ in (3), asymptotic stability was achieved [19] assuming no parametric uncertainty. In the presence of $\mathbf{d_u}$, [20] showed that asymptotic stability is impossible and proposed Uniformly Ultimately Bounded (UUB) stability, again assuming no parametric uncertainty. In the presence of unmatched disturbance and parametric uncertainty, no control design has been reported avoiding the issues in Remark 1, which is the objective of this work.*

Let $\mathbf{q}, \dot{\mathbf{q}}$ be available for feedback, and let us consider a fixed-point equilibrium $\mathbf{q}^d$ for the unforced version of system (5) (without external forces such as inputs and disturbances), i.e., $\mathbf{q}^d$ being feasible. The following is a sufficient condition known in underactuated dynamics literature to ensure controllability to a desired fixed-point equilibrium $\mathbf{q}^d$ of system (5) [29–31].

Assumption 2 *The block $\mathbf{M_{au}}(\mathbf{q})$ is full rank $\forall \mathbf{q} \in \mathbb{R}^n$.*

Remark 4 *Assumption 2 excludes ill-posed cases such as $\mathbf{M_{au}}$ being zero, in which case the non-actuated dynamics (5a) would be uncontrollable. Assumption 2 focuses on stabilization around a fixed point. For tracking of time-varying trajectories, it is challenging to characterize the feasible trajectories: these are in general application-specific (cf. [8, 32] for discussions on feasibility).*

3 Controller Design

Let $\mathbf{q}^d \triangleq [\mathbf{q_a^d}^T \ \mathbf{q_u^d}^T]^T$ and let $\mathbf{e_a} \triangleq \mathbf{q_a} - \mathbf{q_a^d}$, $\mathbf{e_u} \triangleq \mathbf{q_u} - \mathbf{q_u^d}$ be the tracking error in actuated and non-actuated states, respectively. Define an auxiliary error variable $\mathbf{r}$ as:

$$\mathbf{r} \triangleq \mathbf{\Upsilon_a}\dot{\mathbf{e}}_\mathbf{a} + \mathbf{\Gamma_a}\mathbf{e_a} + \mathbf{\Upsilon_u}\dot{\mathbf{e}}_\mathbf{u} + \mathbf{\Gamma_u}\mathbf{e_u}, \tag{6}$$

where $\mathbf{\Upsilon_a}, \mathbf{\Gamma_a} \in \mathbb{R}^{m\times m}$ and $\mathbf{\Upsilon_u}, \mathbf{\Gamma_u} \in \mathbb{R}^{m\times(n-m)}$ are user-defined matrices; $\mathbf{\Upsilon_a}, \mathbf{\Gamma_a}$ are designed such that $\mathbf{\Upsilon_a} > \mathbf{0}, \mathbf{\Gamma_a} > \mathbf{0}$ and $\mathbf{\Upsilon_a}^{-1}\mathbf{\Gamma_a} > \mathbf{0}$; $\mathbf{\Upsilon_u}, \mathbf{\Gamma_u}$ are designed to be of full rank $(n-m)$.

Using (5a) and (5b), the time derivative of (6) yields

$$\begin{aligned}\dot{\mathbf{r}} &= \mathbf{\Upsilon_a}\ddot{\mathbf{q}}_\mathbf{a} + \mathbf{\Gamma_a}\dot{\mathbf{e}}_\mathbf{a} + \mathbf{\Upsilon_u}\ddot{\mathbf{q}}_\mathbf{u} + \mathbf{\Gamma_u}\dot{\mathbf{e}}_\mathbf{u} \\ &= (\mathbf{\Upsilon_a} - \mathbf{\Upsilon_u}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}^{T}})(\mathbf{M_s^{-1}}\boldsymbol{\tau} + \mathbf{h_a}) + \mathbf{\Upsilon_u}\mathbf{h_u} \\ &\quad + \mathbf{\Gamma_a}\dot{\mathbf{e}}_\mathbf{a} + \mathbf{\Gamma_u}\dot{\mathbf{e}}_\mathbf{u} \\ &= \mathbf{b}\boldsymbol{\tau} + \boldsymbol{\phi} + \mathbf{S}_r,\end{aligned} \tag{7}$$

where
$$\begin{aligned}\mathbf{b} &\triangleq (\mathbf{\Upsilon_a} - \mathbf{\Upsilon_u}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}^{T}})\mathbf{M_s^{-1}} \\ \boldsymbol{\phi} &\triangleq (\mathbf{\Upsilon_a} - \mathbf{\Upsilon_u}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}^{T}})\mathbf{h_a} + \mathbf{\Upsilon_u}\mathbf{h_u} \\ \mathbf{S}_r &\triangleq \mathbf{\Gamma_a}\dot{\mathbf{e}}_\mathbf{a} + \mathbf{\Gamma_u}\dot{\mathbf{e}}_\mathbf{u}.\end{aligned}$$

The control law is designed as

$$\boldsymbol{\tau} = \hat{\mathbf{b}}^{-1}(-\mathbf{\Lambda}\mathbf{r} - \mathbf{S}_r - \Delta\boldsymbol{\tau}), \ \Delta\boldsymbol{\tau} = \begin{cases}\rho\frac{\mathbf{r}}{||\mathbf{r}||} & \text{if } ||\mathbf{r}|| \geq \epsilon \\ \rho\frac{\mathbf{r}}{\epsilon} & \text{if } ||\mathbf{r}|| < \epsilon\end{cases}, \tag{8}$$

where $\mathbf{\Lambda} \in \mathbb{R}^{m\times m}$ satisfies $\mathbf{\Lambda} > 0$; $\Delta\boldsymbol{\tau}$ tackles uncertainties utilizing the gain ρ and $\epsilon > 0$ is a small scalar to avoid chattering (the saturation in (8) can be replaced with a smooth sigmoid function with minor modifications in stability analysis [33]). The design of ρ will be discussed later. Finally, $\hat{\mathbf{b}}$ is the nominal value of $\mathbf{b}$ (arising from the nominal knowledge of $\mathbf{M}$) which satisfies:

Assumption 3 *A scalar E is known such that*

$$||\mathbf{b}\hat{\mathbf{b}}^{-1} - \mathbf{I}_m|| \leq E < 1. \tag{9}$$

Remark 5 *Similarly to fully-actuated EL systems [1, 33–35], the value of E can be calculated based on the upper bound of $\Delta\mathbf{M}$ (cf. Assumption 1): the smaller $\Delta\mathbf{M}$, the smaller E, being $E < 1$ required for stability analysis.*

Substituting (8) into (7) yields

$$\dot{\mathbf{r}} = -\mathbf{\Lambda}\mathbf{r} - \Delta\boldsymbol{\tau} + \mathbf{\Psi} - (\mathbf{b}\hat{\mathbf{b}}^{-1} - \mathbf{I}_m)\Delta\boldsymbol{\tau}, \tag{10}$$

where $\mathbf{\Psi} \triangleq \boldsymbol{\phi} - (\mathbf{b}\hat{\mathbf{b}}^{-1} - \mathbf{I}_m)(\mathbf{\Lambda}\mathbf{r} + \mathbf{S}_r)$. From *Properties 1* and *2*, one can verify

$$\begin{aligned}||\mathbf{N}|| &\leq ||\mathbf{C}||||\dot{\mathbf{q}}|| + ||\mathbf{G}|| + ||\mathbf{F}|| \\ &\leq c_b||\dot{\mathbf{q}}||^2 + g_b + f_b||\dot{\mathbf{q}}||.\end{aligned} \tag{11}$$

Let $\boldsymbol{\xi} \triangleq [\mathbf{e}^T \ \dot{\mathbf{e}}^T]^T = [\mathbf{e_a}^T \ \mathbf{e_u}^T \ \dot{\mathbf{e}}_\mathbf{a}^T \ \dot{\mathbf{e}}_\mathbf{u}^T]^T$. Using the fact $||\dot{\mathbf{e}}|| \leq ||\boldsymbol{\xi}||$, and substituting $\dot{\mathbf{q}} = \dot{\mathbf{e}}$ in (11) gives

$$||\mathbf{N}|| \leq g_b + f_b||\boldsymbol{\xi}|| + c_b||\boldsymbol{\xi}||^2. \tag{12}$$

Using the inequalities $||\mathbf{N_u}|| \leq ||\mathbf{N}||, ||\mathbf{N_a}|| \leq ||\mathbf{N}||$, $||\mathbf{d_u}|| \leq ||\mathbf{d}|| \leq \bar{d}, ||\mathbf{d_a}|| \leq ||\mathbf{d}|| \leq \bar{d}$, $||\mathbf{e_a}|| \leq ||\boldsymbol{\xi}||$, $||\mathbf{e_u}|| \leq ||\boldsymbol{\xi}||$, $||\dot{\mathbf{e}}_\mathbf{a}|| \leq ||\boldsymbol{\xi}||$, $||\dot{\mathbf{e}}_\mathbf{u}|| \leq ||\boldsymbol{\xi}||$ in (5) as well as (9) and (12), the following bound can be obtained:

$$\begin{aligned}||\mathbf{\Psi}|| &= ||\boldsymbol{\phi} - (\mathbf{b}\hat{\mathbf{b}}^{-1} - \mathbf{I})(\mathbf{\Lambda}\mathbf{r} + \mathbf{S}_r)|| \\ &\leq ||\boldsymbol{\phi}|| + E(||\mathbf{\Lambda}||||\mathbf{r}|| + ||\mathbf{S}_r||), \\ &\leq \theta_0^* + \theta_1^*||\boldsymbol{\xi}|| + \theta_2^*||\boldsymbol{\xi}||^2,\end{aligned} \tag{13}$$

with

$$\begin{aligned}\theta_0^* &\triangleq a g_b + ||\mathbf{\Upsilon_u}||||\mathbf{M_{uu}^{-1}}||(g_b + \bar{d}) + a_1, \\ \theta_1^* &\triangleq a f_b + ||\mathbf{\Upsilon_u}||||\mathbf{M_{uu}^{-1}}||f_b + E(||\mathbf{\Gamma_a}|| + ||\mathbf{\Gamma_u}||) \\ &\quad + E||\mathbf{\Lambda}||(||\mathbf{\Upsilon_a}|| + ||\mathbf{\Gamma_a}|| + ||\mathbf{\Upsilon_u}|| + ||\mathbf{\Gamma_u}||), \\ \theta_2^* &\triangleq a c_b + ||\mathbf{\Upsilon_u}||||\mathbf{M_{uu}^{-1}}||c_b, \\ a &\triangleq ||(\mathbf{\Upsilon_a} - \mathbf{\Upsilon_u}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}})||(||\mathbf{M_s^{-1}} + ||\mathbf{M_{au}}\mathbf{M_{uu}^{-1}}||), \\ a_1 &\triangleq ||(\mathbf{\Upsilon_a} - \mathbf{\Upsilon_u}\mathbf{M_{uu}^{-1}}\mathbf{M_{au}})||||\mathbf{M_s^{-1}}(1 + ||\mathbf{M_{au}}\mathbf{M_{uu}^{-1}}||)\bar{d},\end{aligned}$$

where the scalars $\theta_i^* \in \mathbb{R}^+$, $i = 0, 1, 2$ are *completely unknown* according to Assumption 1.

Using (5a)-(5b), the error dynamics of the non-actuated dynamics can be represented as

$$\begin{aligned}\ddot{\mathbf{e}}_\mathbf{u} = \ddot{\mathbf{q}}_\mathbf{u} &= -\mathbf{M}_{\mathbf{uu}}^{-1}\mathbf{M}_{\mathbf{au}}^T\ddot{\mathbf{q}}_\mathbf{a} + \mathbf{h}_\mathbf{u} \\ &= -\mathbf{M}_{\mathbf{uu}}^{-1}\mathbf{M}_{\mathbf{au}}^T(\mathbf{M}_\mathbf{s}^{-1}\boldsymbol{\tau} + \mathbf{h}_\mathbf{a}) + \mathbf{h}_\mathbf{u}. \qquad (14)\end{aligned}$$

Substituting (8) into (14) yields

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\mathbf{g}\boldsymbol{\upsilon} - \boldsymbol{\phi}_1, \qquad (15)\end{aligned}$$

where $\mathbf{x}_1 \triangleq \mathbf{e}_\mathbf{u}, \mathbf{x}_2 \triangleq \dot{\mathbf{e}}_\mathbf{u}, \boldsymbol{\phi}_1 \triangleq (\mathbf{M}_{\mathbf{uu}}^{-1}\mathbf{M}_{\mathbf{au}}^T\mathbf{h}_\mathbf{a} + \mathbf{h}_\mathbf{u})$, $\boldsymbol{\upsilon} \triangleq (-\boldsymbol{\Lambda}\mathbf{r} - \mathbf{S}_r - \Delta\boldsymbol{\tau}), \mathbf{g} \triangleq (\mathbf{M}_{\mathbf{uu}}^{-1}\mathbf{M}_{\mathbf{au}}^T\mathbf{M}_\mathbf{s}^{-1})\hat{\mathbf{b}}^{-1}$. Note that $(n-m) \leq m$ by the system definition (1); thus, one can design a constant full-rank matrix $\mathbf{H} \in \mathbb{R}^{(n-m)\times m}$ such that the following holds:

$$\mathbf{K}_1 \triangleq \mathbf{H}\boldsymbol{\Lambda}\boldsymbol{\Gamma}_\mathbf{u} > \mathbf{0}, \ \mathbf{K}_2 \triangleq \mathbf{H}\boldsymbol{\Lambda}\boldsymbol{\Upsilon}_\mathbf{u} > \mathbf{0}. \qquad (16)$$

Adding and subtracting $\mathbf{H}\boldsymbol{\upsilon}$ to (15) yields

$$\begin{aligned}\dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= -\mathbf{K}_1\mathbf{x}_1 - \mathbf{K}_2\mathbf{x}_2 + \mathbf{g}\Delta\boldsymbol{\tau} + \boldsymbol{\phi}_2, \qquad (17)\end{aligned}$$

where $\boldsymbol{\phi}_2 \triangleq \mathbf{g}\mathbf{S}_r + (\mathbf{H}+\mathbf{g})\boldsymbol{\Lambda}\mathbf{r} - \boldsymbol{\phi}_1 - \mathbf{H}\boldsymbol{\Lambda}(\boldsymbol{\Upsilon}_\mathbf{a}\dot{\mathbf{e}}_\mathbf{a} + \boldsymbol{\Gamma}_\mathbf{a}\mathbf{e}_\mathbf{a})$. In line with [29–31] Assumption 2 is required because, when $\mathbf{M}_{\mathbf{au}}$ is not full rank, the rank deficiency in $\mathbf{g}$ would not allow to tackle the uncertainty $\boldsymbol{\phi}_2$ via $\Delta\boldsymbol{\tau}$.

Therefore, taking $\mathbf{x} \triangleq [\mathbf{x}_1^T \ \mathbf{x}_2^T]^T$, $\mathbf{A} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I}_{(n-m)} \\ -\mathbf{K}_1 & -\mathbf{K}_2 \end{bmatrix}$ and $\mathbf{B} \triangleq \begin{bmatrix}\mathbf{0} \ \mathbf{I}_{(n-m)}\end{bmatrix}^T$, one has from (17)

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}(\mathbf{g}\Delta\boldsymbol{\tau} + \boldsymbol{\phi}_2) \qquad (18)$$

where $\mathbf{K}_1 > \mathbf{0}, \mathbf{K}_2 > \mathbf{0}$ guarantee that $\mathbf{A}$ is Hurwitz. From *Properties 1-2*, the following holds

$$||\boldsymbol{\phi}_2||||\mathbf{PB}|| \leq \left(\theta_0^{**} + \theta_1^{**}||\boldsymbol{\xi}|| + \theta_2^{**}||\boldsymbol{\xi}||^2\right), \qquad (19)$$

where $\theta_i^{**} \in \mathbb{R}^+$, $i = 0, 1, 2$ are unknown scalars whose expressions follow from similar steps as (13); $\mathbf{P} > \mathbf{0}$ is the solution to the Lyapunov equation $\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} = -\mathbf{Q}$ for some $\mathbf{Q} > \mathbf{0}$. The term $||\mathbf{PB}||$ in (19) is primarily considered for subsequent mathematical simplifications. The vectors $\boldsymbol{\Psi}$ and $\boldsymbol{\phi}_2$ act as the *overall uncertainty* in the closed-loop dynamics (7) and (18), respectively.

Remark 6 *A standard assumption in underactuated EL literature is that the system dynamics terms are LIP [12, 13, 23–25]. Control designs for NLIP systems have appeared, e.g., in [36], which however do not consider underactuation. It is noteworthy that, due to Properties 1 and 2, the upper bounds of* $||\boldsymbol{\Psi}||$ *and* $||\boldsymbol{\phi}_2||$ *always exhibit the LIP structures (13) and (19), irrespective of the fact that the terms* $\boldsymbol{\Psi}$ *and* $\boldsymbol{\phi}_2$ *are LIP or NLIP.*

Using the structures of the upper bounds of $||\boldsymbol{\Psi}||$ and $||\boldsymbol{\phi}_2||$ in (13) and (19) respectively, we are now ready to design ρ in (8) as

$$\rho = \frac{1}{(1-E)}(\hat{\theta}_0 + \hat{\theta}_1||\boldsymbol{\xi}|| + \hat{\theta}_2||\boldsymbol{\xi}||^2 + \gamma), \qquad (20)$$

with adaptive laws ($i = 0, 1, 2$)

$$\begin{aligned}\dot{\hat{\theta}}_i &= \eta_i(||\mathbf{r}|| + ||\mathbf{x}||)||\boldsymbol{\xi}||^i - \zeta_i\hat{\theta}_i\beta||\mathbf{x}||||\boldsymbol{\xi}||^i, \qquad (21a)\\ \dot{\gamma} &= -\gamma\left\{\gamma_0 + \gamma_1(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) + \gamma_2(||\mathbf{x}|| + ||\boldsymbol{\xi}||)\right\} \\ &\quad + \gamma_0(||\mathbf{r}|| + ||\mathbf{x}||) + \gamma_0\nu, \qquad (21b)\\ &\text{initial conditions } \hat{\theta}_i(0) > 0, \ \gamma(0) > \nu, \qquad (21c)\\ &\text{and } \eta_i, \zeta_i, \beta, \gamma_0, \gamma_1, \gamma_2, \nu \in \mathbb{R}^+, \qquad (21d)\end{aligned}$$

satisfying the following inequalities

$$\gamma_2 \geq \gamma_1, \ \beta > 1 + (E_1/(1-E)), \qquad (21e)$$

with E_1 being a constant satisfying $||\mathbf{PBg}|| \leq E_1$, and derived from the known upper bound of $\Delta\mathbf{M}$ in Assumption 1. In (21), $\hat{\theta}_i$ is the estimate of $\bar{\theta}_i^* \triangleq \max\{\theta_i^*, \theta_i^{**}\}$, $i = 0, 1, 2$; γ is an auxiliary gain which has a crucial role in closed-loop system stabilization and it will be detailed later (cf. Remark 7). It can be verified that the design $\gamma_0, \gamma_1, \gamma_2 \in \mathbb{R}^+$ with $\gamma_2 \geq \gamma_1$ makes the term '$\gamma_0 + \gamma_1(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) + \gamma_2(||\mathbf{x}|| + ||\boldsymbol{\xi}||)$' in (21b) positive for all $\mathbf{x}, \boldsymbol{\xi}$.

4 Stability Analysis of The Proposed ARC

Theorem 4 *Under Properties 1-2 and Assumptions 1-3, the closed-loop trajectories of (5) employing the ARC laws (8), (20) with gain conditions (16) and adaptive laws (21) are Uniformly Ultimately Bounded (UUB).*

Proof. Stability is analyzed via the Lyapunov function:

$$V = \frac{1}{2}\left\{\mathbf{r}^T\mathbf{r} + \mathbf{x}^T\mathbf{P}\mathbf{x} + \sum_{i=0}^{2}\frac{1}{\eta_i}(\hat{\theta}_i - \bar{\theta}_i^*)^2 + \frac{\gamma^2}{\gamma_0}\right\}, \qquad (22)$$

where $\bar{\theta}_i^* = \max\{\theta_i^*, \theta_i^{**}\}$. Stability analysis considers the two cases (i) $||\mathbf{r}|| \geq \epsilon$ and (ii) $||\mathbf{r}|| < \epsilon$ using the common Lyapunov function (22).

Case (i) $||\mathbf{r}|| \geq \epsilon$

Using (8), (13) and (20), from (7) we have

$$\begin{aligned}\mathbf{r}^T\dot{\mathbf{r}} &= \mathbf{r}^T(-\mathbf{\Lambda}\mathbf{r} - \Delta\boldsymbol{\tau} + \mathbf{\Psi} - (\mathbf{b}\hat{\mathbf{b}}^{-1} - \mathbf{I}_m)\Delta\boldsymbol{\tau}) \\ &\leq -\mathbf{r}^T\mathbf{\Lambda}\mathbf{r} - (1-E)\rho||\mathbf{r}|| + \sum_{i=0}^{2}\theta_i^*||\boldsymbol{\xi}||^i||\mathbf{r}|| \\ &\leq -\mathbf{r}^T\mathbf{\Lambda}\mathbf{r} - \sum_{i=0}^{2}(\hat{\theta}_i||\boldsymbol{\xi}||^i + \gamma)||\mathbf{r}|| + \bar{\theta}_i^*||\boldsymbol{\xi}||^i||\mathbf{r}||. \quad (23)\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^T\mathbf{P}\mathbf{x} &= -\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{x}^T\mathbf{P}\mathbf{B}(\mathbf{g}\Delta\boldsymbol{\tau} + \boldsymbol{\phi}_2) \\ &\leq -\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \rho E_1||\mathbf{x}|| + ||\boldsymbol{\phi}_2||||\mathbf{P}\mathbf{B}||||\mathbf{x}||. \quad (24)\end{aligned}$$

Substituting (19) into (24) yields

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{x}^T\mathbf{P}\mathbf{x} &\leq -\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} + \sum_{i=0}^{2}\theta_i^*||\boldsymbol{\xi}||^i||\mathbf{x}|| \\ &+ \frac{E_1}{1-E}\sum_{i=0}^{2}(\hat{\theta}_i||\boldsymbol{\xi}||^i + \gamma)||\mathbf{x}|| \leq -\frac{1}{2}\mathbf{x}^T\mathbf{Q}\mathbf{x} \\ &+ \sum_{i=0}^{2}\bar{\theta}_i^*||\boldsymbol{\xi}||^i||\mathbf{x}|| + \frac{E_1}{1-E}\sum_{i=0}^{2}(\hat{\theta}_i||\boldsymbol{\xi}||^i + \gamma)||\mathbf{x}||. \quad (25)\end{aligned}$$

Using the adaptive laws (21a) and (21b), we have

$$\begin{aligned}(1/\eta_i)(\hat{\theta}_i - \bar{\theta}_i^*)\dot{\hat{\theta}}_i &= (1/\eta_i)(\hat{\theta}_i - \bar{\theta}_i^*)(\eta_i(||\mathbf{r}|| + ||\mathbf{x}||)||\boldsymbol{\xi}||^i \\ &\quad - \zeta_i\hat{\theta}_i\beta||\mathbf{x}||||\boldsymbol{\xi}||^i) \\ &= \hat{\theta}_i(||\mathbf{r}|| + ||\mathbf{x}||)||\boldsymbol{\xi}||^i - c_i\hat{\theta}_i^2||\mathbf{x}||||\boldsymbol{\xi}||^i \\ &\quad - \bar{\theta}_i^*(||\mathbf{r}|| + ||\mathbf{x}||)||\boldsymbol{\xi}||^i + c_i\hat{\theta}_i\bar{\theta}_i^*||\mathbf{x}||||\boldsymbol{\xi}||^i, \quad (26) \\ \frac{\gamma\dot{\gamma}}{\gamma_0} &= \frac{\gamma}{\gamma_0}\{\gamma_0(||\mathbf{r}|| + ||\mathbf{x}||) - \gamma(\gamma_0 + \gamma_1(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) \\ &\quad + \gamma_2||\mathbf{x}||) + \gamma_0\nu\} \\ &= \gamma(||\mathbf{r}|| + ||\mathbf{x}||) - \gamma^2\{1 + \bar{\gamma}(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) \\ &\quad + c_3(||\mathbf{x}|| + ||\boldsymbol{\xi}||)\} + \gamma\nu, \quad (27)\end{aligned}$$

for $i = 0, 1, 2$, where $c_i \triangleq \frac{\zeta_i}{\eta_i}\beta$, $c_3 \triangleq \frac{\gamma_2}{\gamma_0}$ and $\bar{\gamma} \triangleq \frac{\gamma_1}{\gamma_0}$ are positive by virtue of the choice of (21d)-(21e). Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{i=0}^{2}\frac{(\hat{\theta}_i - \bar{\theta}_i^*)^2}{2\eta_i} + \frac{\gamma^2}{2\gamma_0}\right) &= \sum_{i=0}^{2}\hat{\theta}_i(||\mathbf{r}|| + ||\mathbf{x}||)||\boldsymbol{\xi}||^i \\ - c_i\hat{\theta}_i^2||\mathbf{x}||||\boldsymbol{\xi}||^i - \bar{\theta}_i^*(||\mathbf{r}|| + ||\mathbf{x}||)&||\boldsymbol{\xi}||^i + c_i\hat{\theta}_i\bar{\theta}_i^*||\mathbf{x}||||\boldsymbol{\xi}||^i \\ - \gamma^2\{1 + \bar{\gamma}(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) &+ c_3(||\mathbf{x}|| + ||\boldsymbol{\xi}||))\} \\ + \gamma(||\mathbf{r}|| + ||\mathbf{x}||) + \gamma\nu. &\quad (28)\end{aligned}$$

Using (23), (25) and (28), the time derivative of the Lyapunov function (22) turns out to be

$$\begin{aligned}\dot{V} \leq &- \delta_m(||\mathbf{r}||^2 + ||\mathbf{x}||^2) + \gamma\nu + c\gamma||\mathbf{x}|| \\ &- \gamma^2\{1 + \bar{\gamma}(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) + c_3||\mathbf{x}||\} \\ &+ \sum_{i=0}^{2}(c\hat{\theta}_i - c_i\hat{\theta}_i^2 + c_i\hat{\theta}_i\bar{\theta}_i^*)||\boldsymbol{\xi}||^i||\mathbf{x}||, \quad (29)\end{aligned}$$

where $\delta_m \triangleq \min\{\lambda_{\min}(\mathbf{\Lambda}),\ (1/2)\lambda_{\min}(\mathbf{Q})\}$ and $c \triangleq 1 + \frac{E_1}{1-E}$. From $\hat{\theta}_i(t) \geq 0$, the definition of V in (22) yields

$$V \leq \delta_M(||\mathbf{r}||^2 + ||\mathbf{x}||^2) + \sum_{i=0}^{2}\frac{(\hat{\theta}_i^2 + \bar{\theta}_i^{*2})}{\eta_i} + \frac{\gamma^2}{\gamma_0}, \quad (30)$$

where $\delta_M \triangleq \max\{1,\ ||\mathbf{P}||\}$. Defining $\Omega \triangleq (\delta_m/\delta_M)$ and using (30), inequality (29) is further simplified to

$$\begin{aligned}\dot{V} \leq &- \Omega V + \sum_{i=0}^{2}\frac{\Omega}{\eta_i}(\hat{\theta}_i^2 + \bar{\theta}_i^*)^2 + \frac{\Omega}{\gamma_0}\gamma^2 \\ &+ \sum_{i=0}^{2}(c\hat{\theta}_i - c_i\hat{\theta}_i^2 + c_i\hat{\theta}_i\bar{\theta}_i^*)||\boldsymbol{\xi}||^i||\mathbf{x}|| + \gamma\nu + c\gamma||\mathbf{x}|| \\ &- \gamma^2(1 + \bar{\gamma}(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) + c_3||\mathbf{x}||). \quad (31)\end{aligned}$$

Since c_i and c_3 are positive constants by design, it is always possible to split these terms as

$$c_i = \sum_{j=1}^{3}c_{ij},\ c_3 = \sum_{k=1}^{2}c_{3k},\quad c_{ij}, c_{3k} > 0\ \forall i,j,k, \quad (32)$$

leading to the following simplifications

$$\begin{aligned}&- c_i\hat{\theta}_i^2 + c\hat{\theta}_i + c_i\hat{\theta}_i\bar{\theta}_i^* \\ &= -c_{i1}\hat{\theta}_i^2 - c_{i2}\left\{\left(\hat{\theta}_i - (c/(2c_{i2}))\right)^2 - (c^2/(4c_{i2}^2))\right\} \\ &\quad - c_{i3}\left\{\left(\hat{\theta}_i - ((c_i\bar{\theta}_i^*)/(2c_{i3}))\right)^2 - ((c_i\bar{\theta}_i^*)^2/(4c_{i3}^2))\right\} \\ &\leq -c_{i1}\hat{\theta}_i^2 + c^2/(4c_{i2}) + (c_i\bar{\theta}_i^*)^2/(4c_{i3}). \quad (33)\end{aligned}$$

Further,

$$\begin{aligned}&- \gamma^2(1 + c_3||\mathbf{x}||) + \gamma\nu + c\gamma||\mathbf{x}|| \\ &= -c_{31}\gamma^2||\mathbf{x}|| - \left\{(\gamma - (\nu/2))^2 - (\nu/2)^2\right\} \\ &\quad - c_{32}||\mathbf{x}||\left\{(\gamma - (c/2c_{32}))^2 - \left(c^2/4c_{32}^2\right)\right\} \\ &\leq -c_{31}\gamma^2||\mathbf{x}|| + \left(c^2/4c_{32}\right)||\mathbf{x}|| + (\nu^2/4). \quad (34)\end{aligned}$$

Investigating the adaptive laws (21a)-(21b) (first-order time-varying linear systems with negative system matrix and positive input) and the initial conditions (21c), it can be verified that $\hat{\theta}_i(t) \geq 0$ and $\gamma(t) \geq \underline{\gamma} > 0\ \ \forall t \geq 0$

for some positive scalar $\underline{\gamma}$. Further $||\boldsymbol{\xi}|| \geq ||\mathbf{x}||$. Then, using (33)-(34), inequality (31) becomes

$$\begin{aligned}\dot{V} \leq & -\Omega V + \sum_{i=0}^{2}(\Omega/\eta_i)(\hat{\theta}_i^2 + (\bar{\theta}_i^*)^2) + (\Omega/\gamma_0)\gamma^2 + (\nu^2/4) \\ & + \sum_{i=0}^{2}\left(\frac{c^2}{4c_{i2}} + \frac{\left(c_i\bar{\theta}_i^*\right)^2}{4c_{i3}}\right)||\boldsymbol{\xi}||^{i+1} - c_{i1}\hat{\theta}_i^2||\mathbf{x}||^{i+1} \\ & - \underline{\gamma}^2\bar{\gamma}(||\boldsymbol{\xi}||^5 - ||\boldsymbol{\xi}||^4) - c_{31}\gamma^2||\mathbf{x}|| + \left(c^2/4c_{32}\right)||\mathbf{x}|| \\ = & -\Omega V - \hat{\theta}_0^2\left(c_{01}||\mathbf{x}|| - (\Omega/\eta_0)\right) + f(||\boldsymbol{\xi}||) \\ & - \hat{\theta}_1^2\left(c_{11}||\mathbf{x}||^2 - (\Omega/\eta_1)\right) - \hat{\theta}_2^2\left(c_{21}||\mathbf{x}||^3 - (\Omega/\eta_2)\right) \\ & - \gamma^2\left(c_{31}||\mathbf{x}|| - (\Omega/\gamma_0)\right), \end{aligned} \quad (35)$$

where

$$\begin{aligned} f(||\boldsymbol{\xi}||) &\triangleq -\underline{\gamma}^2\bar{\gamma}||\boldsymbol{\xi}||^5 + \omega_4||\boldsymbol{\xi}||^4 + \omega_3||\boldsymbol{\xi}||^3 + \omega_2||\boldsymbol{\xi}||^2 + \omega_1||\boldsymbol{\xi}|| + \omega_0, \\ \omega_3 &\triangleq c^2/(4c_{22}) + \left(\left(c_2\bar{\theta}_2^*\right)^2/(4c_{23})\right), \omega_4 \triangleq \underline{\gamma}^2\bar{\gamma} \\ \omega_2 &\triangleq \frac{c^2}{4c_{12}} + \frac{\left(c_1\bar{\theta}_1^*\right)^2}{4c_{13}}, \omega_1 \triangleq \frac{c^2}{4c_{02}} + \frac{\left(c_0\bar{\theta}_0^*\right)^2}{4c_{03}} + \frac{c^2}{4c_{32}}, \\ \omega_0 &\triangleq \sum_{i=0}^{2}(\Omega/\eta_i)(\bar{\theta}_i^*)^2 + (\nu^2/4). \end{aligned}$$

Using Descartes' rule of sign change [37] and Bolzano's Theorem [38], it can be verified that the polynomial f has exactly one positive real root. Let $\iota \in \mathbb{R}^+$ be the positive real root of f. The coefficient of the highest degree of f is negative as $\underline{\gamma}^2\bar{\gamma} \in \mathbb{R}^+$. Therefore, $f(||\boldsymbol{\xi}||) \leq 0$ when $||\boldsymbol{\xi}|| \geq \iota$. Define $\iota_0 \triangleq \frac{\Omega}{\eta_0 c_{01}}, \iota_1 \triangleq \sqrt{\frac{\Omega}{\eta_1 c_{11}}}, \iota_2 \triangleq \left(\frac{\Omega}{\eta_2 c_{21}}\right)^{1/3}$ and $\iota_3 \triangleq \frac{\Omega}{\gamma_0 c_{31}}$. Hence, from (35), $\dot{V} \leq -\Omega V$ when

$$\begin{aligned} \min\{||\mathbf{x}||, ||\boldsymbol{\xi}||\} &\geq \max\{\iota,\ \iota_0,\ \iota_1,\ \iota_2,\ \iota_3\} \\ \Rightarrow ||\mathbf{x}|| &\geq \max\{\iota,\ \iota_0,\ \iota_1,\ \iota_2,\ \iota_3\}. \end{aligned} \quad (36)$$

Case (ii) $||\mathbf{r}|| < \epsilon$

Using (8), (13) and (20), from (7), for $||\mathbf{r}|| < \epsilon$ we have

$$\begin{aligned} \mathbf{r}^T\dot{\mathbf{r}} &\leq -\mathbf{r}^T\boldsymbol{\Lambda}\mathbf{r} - (1-E)\rho(||\mathbf{r}||^2/\epsilon) + \sum_{i=0}^{2}\theta_i^*||\boldsymbol{\xi}||^i||\mathbf{r}|| \\ &\leq -\mathbf{r}^T\boldsymbol{\Lambda}\mathbf{r} + \sum_{i=0}^{2}\bar{\theta}_i^*||\boldsymbol{\xi}||^i||\mathbf{r}||. \end{aligned} \quad (37)$$

Before proceeding further, the following simplification is made for $i = 0, 1, 2$:

$$\begin{aligned} \epsilon\hat{\theta}_i||\boldsymbol{\xi}||^i &= \hat{\theta}_i^2 - \left\{\left(\hat{\theta}_i - (\epsilon||\boldsymbol{\xi}||^i)/2\right)^2 - (\epsilon^2||\boldsymbol{\xi}||^{(2i)})/4\right\} \\ &\leq \hat{\theta}_i^2 + (\epsilon^2||\boldsymbol{\xi}||^{(2i)})/4. \end{aligned} \quad (38)$$

Using (37)-(38) and a similar procedure as Case (i), the following can be deduced for Case (ii):

$$\begin{aligned} \dot{V} \leq & -\Omega V - \hat{\theta}_0^2\left(c_{01}||\mathbf{x}|| - ((\Omega/\eta_0)+1)\right) + f_1(||\boldsymbol{\xi}||) \\ & - \hat{\theta}_1^2\left(c_{11}||\mathbf{x}||^2 - ((\Omega/\eta_1)+1)\right) \\ & - \hat{\theta}_2^2\left(c_{21}||\mathbf{x}||^3 - ((\Omega/\eta_2)+1)\right) \\ & - \gamma^2\left(c_{31}||\mathbf{x}|| - ((\Omega/\gamma_0)+1)\right), \end{aligned} \quad (39)$$

where

$$\begin{aligned} f_1(||\boldsymbol{\xi}||) &\triangleq -\underline{\gamma}^2\bar{\gamma}||\boldsymbol{\xi}||^5 + \omega_4'||\boldsymbol{\xi}||^4 + \omega_3||\boldsymbol{\xi}||^3 + \omega_2'||\boldsymbol{\xi}||^2 + \omega_1||\boldsymbol{\xi}|| + \omega_0', \\ \omega_4' &\triangleq \omega_4 + (\epsilon^2/4), \omega_2' \triangleq \omega_2 + (\epsilon^2/4) \\ \omega_0' &\triangleq \sum_{i=0}^{2}(\Omega/\eta_i)(\bar{\theta}_i^*)^2 + ((\nu+\epsilon)^2/4) + (\epsilon^2/4). \end{aligned}$$

Asserting similar argument made for Case (i), $\dot{V} \leq -\Omega V$ is guaranteed when

$$||\mathbf{x}|| \geq \max\{\iota',\ \iota_0',\ \iota_1',\ \iota_2',\ \iota_3'\}, \quad (40)$$

where ι' is the sole positive real root of the polynomial f_1 and $\iota_0' \triangleq \left(\frac{\Omega}{\eta_0 c_{01}} + (1/c_{01})\right), \iota_1' \triangleq \sqrt{\frac{\Omega}{\eta_1 c_{11}} + \frac{1}{c_{11}}}, \iota_2' \triangleq \left(\frac{\Omega}{\eta_2 c_{21}} + \frac{1}{c_{21}}\right)^{1/3}$ and $\iota_3' \triangleq \left(\frac{\Omega}{\gamma_0 c_{31}} + \frac{1}{c_{31}}\right)$.

Hence, investigating the results for Cases (i) and (ii), UUB stability can be concluded, implying $\mathbf{r}, \mathbf{e_u}, \dot{\mathbf{e}}_\mathbf{u}, \hat{\theta}_i, \gamma \in \mathcal{L}_\infty$. Let us now write (6) as

$$\dot{\mathbf{e}}_\mathbf{a} = -\boldsymbol{\Upsilon}_\mathbf{a}^{-1}\boldsymbol{\Gamma}_\mathbf{a}\mathbf{e_a} - \boldsymbol{\Upsilon}_\mathbf{a}^{-1}(\boldsymbol{\Upsilon}_\mathbf{u}\dot{\mathbf{e}}_\mathbf{u} + \boldsymbol{\Gamma}_\mathbf{u}\mathbf{e_u}) + \boldsymbol{\Upsilon}_\mathbf{a}^{-1}\mathbf{r} \quad (41)$$

where $\boldsymbol{\Upsilon}_\mathbf{a}^{-1}$ exists being $\boldsymbol{\Upsilon}_\mathbf{a} > \mathbf{0}$. Using $\boldsymbol{\Upsilon}_\mathbf{a}^{-1}\boldsymbol{\Gamma}_\mathbf{a} > \mathbf{0}$ and $\mathbf{r}, \mathbf{e_u}, \dot{\mathbf{e}}_\mathbf{u} \in \mathcal{L}_\infty$ we conclude that $\mathbf{e_a}, \dot{\mathbf{e}}_\mathbf{a} \in \mathcal{L}_\infty$. This concludes the proof. Note that the leakage action in (21a) and (21b) cannot guarantee convergent error even with convergent disturbance, a standard result in adaptive robust control [5,34,35]. □

Remark 7 *The importance of the auxiliary gain γ in (8) can be realized from (35) and (39). There, the negative fifth degree term $-\underline{\gamma}^2\bar{\gamma}||\boldsymbol{\xi}||^5$ (contributed by $\dot{\gamma}$) ensures stability by making $f(||\boldsymbol{\xi}||) \leq 0$ for $||\boldsymbol{\xi}|| \geq \iota$ and $f_1(||\boldsymbol{\xi}||) \leq 0$ for $||\boldsymbol{\xi}|| \geq \iota'$ for Case (i) and (ii).*

Remark 8 *The behavior of the non-actuated dynamics (17) can be tuned via $\mathbf{K}_1, \mathbf{K}_2$, designed through (16); (41) reveals that large $\boldsymbol{\Upsilon}_\mathbf{a}^{-1}\boldsymbol{\Gamma}_\mathbf{a}$ leads to faster convergence of actuated errors. Large values of $\eta_i, \gamma_1, \gamma_2$ and ν lead to large $\underline{\gamma}^2\bar{\gamma}$ and small ι_i, ι_i', making the polynomials $f(||\boldsymbol{\xi}||) < 0$ and $f_1(||\boldsymbol{\xi}||) < 0$ (cf. the definition of $\bar{\gamma}$ after (27) and the negative fifth-degree term in (35) and (39)). However, such large values might lead to high control input and therefore these gains should be selected according to application requirements.*

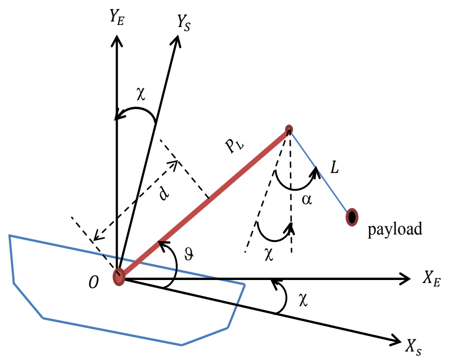

Figure 1. A schematic diagram of offshore boom crane.

5 Simulation Results

To evaluate the effectiveness of the proposed ARC, an offshore boom crane example is considered. In Fig. 1, $\{OX_EY_E\}$ and $\{OX_sY_s\}$ define the Earth-fixed and ship-fixed coordinates, respectively; ϑ is the luffing angle of the boom; α is the swing with respect to Y_s of the payload having mass m_p; χ is roll angle of the ship caused by sea wave/current; $L(t)$ is the length of the rope; P_L, m and J are the length, mass and inertia of the boom; and d is the distance between the barycenter of the boom and the point O. The objective is to take the payload to a target location (a_L, b_L) (in the Earth-fixed frame) while rejecting the ship rolling. To this end, denoting $q_1 = \vartheta - \chi, q_2 = L$, and $q_3 = \alpha - \chi$, the dynamics of the offshore boom crane as in Fig. 1 is given as [11]:

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q}, \dot{\mathbf{q}})\dot{\mathbf{q}} + \mathbf{G}(\mathbf{q}) + \mathbf{d_s} = [\boldsymbol{\tau}^T \ 0]^T, \qquad (42)$$

where

$$\mathbf{M}(\mathbf{q}) = \begin{bmatrix} J + m_p P_L^2 & -m_p P_L C_{1-3} & -m_p P_L q_2 S_{1-3} \\ -m_p P_L C_{1-3} & m_p & 0 \\ -m_p P_L q_2 S_{1-3} & 0 & m_p q_2^2 \end{bmatrix}$$

$$\mathbf{C}(\mathbf{q}, \dot{\mathbf{q}}) = \begin{bmatrix} 0 & -m_p P_L S_{1-3}\dot{q}_3 & \vartheta_{13} \\ m_p P_L S_{1-3}\dot{q}_1 & 0 & -m_p q_2 \dot{q}_3 \\ -m_p P_L q_2 C_{1-3}\dot{q}_1 & m_p q_2 \dot{q}_3 & m_p q_2 \dot{q}_2 \end{bmatrix}$$

$$\vartheta_{13} = -m_p P_L (S_{1-3}\dot{q}_2 - C_{1-3} q_2 \dot{q}_3),$$

$$\mathbf{G}(\mathbf{q}) = \begin{bmatrix} (m_p P_L + md) g_a \cos(q_1) \\ -m_p g_a \cos(q_3) \\ m_p g_a q_2 \sin(q_3) \end{bmatrix}, \boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}, \mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix},$$

where $S_{1-3} \triangleq \sin(q_1 - q_3), C_{1-3} \triangleq \cos(q_1 - q_3)$; g_a is the gravity constant; $\mathbf{d_s}$ are external disturbances representing the effects of wind, slack in crane wires etc. We select $\mathbf{d_s} = (0.1\sin(0.01t) + d_n)\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^T$ with d_n a zero-mean

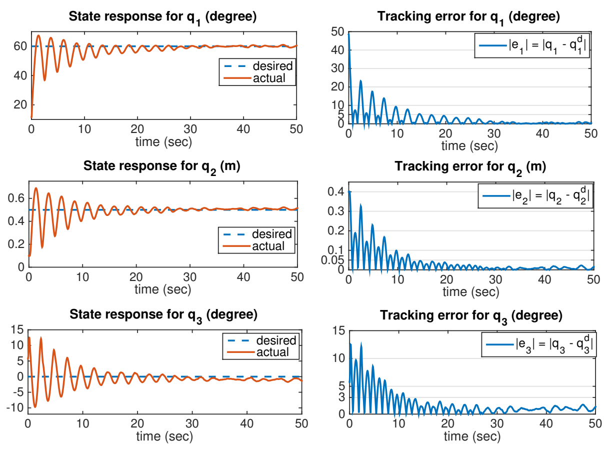

Figure 2. The tracking performance of ARC.

Gaussian noise with variance 0.002. Note that *the symmetry condition of [7, 17–20] does not hold for (42)*, as $\mathbf{M}$ depends on both the actuated (q_1, q_2) and non-actuated (q_3) states. Transporting the payload to (a_L, b_L) can be transformed into stabilization around the following desired position (cf. [11] for the derivation)

$$q_1^d = \arccos(a_L/P_L),\ q_2^d = \sqrt{P_L^2 - a_L^2} - b_L,\ q_3^d = 0.$$

i.e. the control problem for (42) is well posed. Note that no uncertainty in the length of the boom P_L must be considered in order to calculate the desired equilibrium. In the simulations we take $a_L = 0.4$ m, $b_L = 0.2$ m and $P_L = 0.8$ m, resulting in $q_1^d = 1.05$ rad (60 degrees) and $q_2^d = 0.5$ m (all numerical values are appropriately scaled as in [11] and they are not representative of a real crane).

The actual (and uncertain) system parameters are selected as: $m = 20$ kg, $m_p = 0.5$ kg, $d = 0.4$ m and $J = 6.5$ kg-m^2. The nominal parameters (to compute $\hat{\mathbf{b}}$) are selected as $\hat{m}_p = 0.45$ kg and $\hat{J} = 6$ kg-m^2. By substituting the actual and nominal values in the mass matrix, it is possible to see that Assumption 3 is satisfied with $E = 0.5$. The parametric values of m and d and the upper bound of $\mathbf{d_s}$ are considered to be unknown. The initial configuration is selected as $(q_1(0), q_2(0), q_3(0)) = (0.2, 0.1, 0.1)$. The control design parameters are $\boldsymbol{\Upsilon}_\mathbf{a} = 50\mathbf{I}_2, \boldsymbol{\Upsilon}_\mathbf{u} = 50[1\ 1]^T, \boldsymbol{\Gamma}_\mathbf{a} = 400\mathbf{I}_2, \boldsymbol{\Gamma}_\mathbf{u} = 400[1\ 1]^T, \boldsymbol{\Lambda} = 15\mathbf{I}_2, \epsilon = 1, \mathbf{H} = [1\ 1]^T, \mathbf{Q} = \mathbf{I}_2, \eta_0 = 3, \eta_1 = 4, \eta_2 = 1, \zeta_0 = 1, \zeta_1 = 4, \zeta_2 = 1, \gamma_0 = 2, \gamma_1 = \gamma_2 = \nu = 1, \hat{\theta}_i(0) = \gamma(0) = 10, i = 0, 1, 2$. The solution $\mathbf{P}$ to the Lyapunov equation after (19) and the upper bound of $\Delta\mathbf{M}$ results in $E_1 = 1$, and thus $\beta = 3$.

The tracking performance of the proposed ARC is provided in Fig. 2 in terms of state responses and tracking error (absolute value) for both actuated and non-actuated states (the plots are given in degrees for better inference). Therein, after $t = 20$sec, the tracking errors

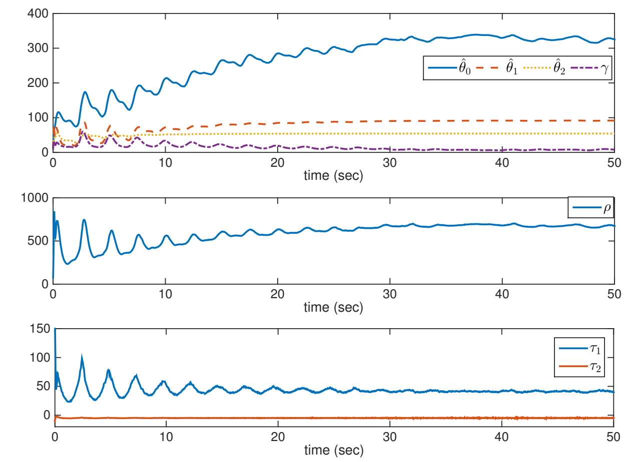

Figure 3. The evaluations of various gains and control input.

in q_1 and q_3 are below 5 and 3 degrees respectively, while for q_2 the error is below 0.05m. Fig. 3 demonstrates the (bounded) evolutions of the various gains $\hat{\theta}_i, \gamma$, the overall gain ρ and control input τ.

6 Conclusions

An adaptive robust controller was proposed for a class of uncertain underactuated Euler-Lagrange systems. Compared to the existing methodologies, the proposed one avoids structure-specific restrictions such as symmetry condition of the mass matrix, and a priori bounds on non-actuated states or on state derivatives. Also, the requirement of knowing dynamics terms such as Coriolis, centripetal, friction and gravity terms has been removed, while state-dependent uncertainties have been handled irrespective of their linear or nonlinear in parameters structure.

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