A note on robustness of coherence for multipartite quantum states

by

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I. INTRODUCTION

Quantum Mechanics is the theoretical cornerstone underpinning our understanding of the natural world. The abstract laws of quantum mechanics also present us with resources we can harness to perform practical and important information theoretic tasks [1]. Motivated by the importance of quantum entanglement [2] in quantum communication schemes, a general study of the theory of resources within the quantum framework and beyond is being developed at present. One such concrete example of a quantum resource theory is the resource theory of quantum coherence, including two most popular measures of quantum coherence derived from the resource theoretic framework of coherence, may be sub-additive for a specific class of multipartite quantum states. We investigate how the sub-additivity is affected by admixture with other classes of states for which ROC is super-additive. We show that pairs of quantum states may have different orderings with respect to relative entropy of coherence, $l_1$-norm of coherence and ROC and numerically study the difference in ordering for coherence measures chosen pairwise.

II. ROBUSTNESS OF COHERENCE

At first, we shall look into the criteria needed by a functional $C$ to qualify as a measure of coherence. In Baumgratz’s framework [4], a functional $C$, mapping quantum states to a non-negative real numbers, must satisfy the following properties to qualify as a measure of quantum coherence:

- [C1] Firstly, $C$ should vanish on all incoherent states: $C(\rho) = 0$ for all incoherent states $\rho$.

In this brief report, we prove that robustness of coherence (ROC), in contrast to many popular quantitative measures of quantum coherence derived from the resource theoretic framework of coherence, may be sub-additive for a specific class of multipartite quantum states. We investigate how the sub-additivity is affected by admixture with other classes of states for which ROC is super-additive. We show that pairs of quantum states may have different orderings with respect to relative entropy of coherence, $l_1$-norm of coherence and ROC and numerically study the difference in ordering for coherence measures chosen pairwise.
\[ C(\rho) = 0, \forall \rho \in I, \text{ where } I \text{ is the set of all incoherent states in the given basis.} \]

- **[C2]** Secondly, \( C \) should not increase under incoherent operations, which can be of types A and B.
  
  - **[C2a]** Under type A operations, we have monotonicity under incoherent completely positive and trace preserving maps, that is, \( C(\rho) \geq C(\Phi_{\text{CPTP}}(\rho)), \forall \Phi_{\text{CPTP}}. \)
  
  - **[C2b]** Under type B operations, we have monotonicity under selective measurements on average, that is, \( C(\rho) \geq \sum_{n} p_n C(\rho_n), \forall \{K_n\} \text{ such that } \sum_n K_n^* K_n = I \text{ and } K_n K_m^* \subset I, \text{ where } I \text{ is the set of all incoherent states in the given basis.} \)

- **[C3]** Moreover, we would ideally like to ensure that coherence can only decrease under mixing, which leads to our final condition: non-increasing under mixing of quantum states (convexity), that is, \( C(\rho) \leq C(\sum_n p_n \rho_n) \) for any set of states \( \{\rho_n\} \) and any \( p_n \geq 0 \) with \( \sum p_n = 1. \)

Now, we recall the definition of Robustness of coherence which satisfies all the above criteria for a coherence monotone.

**Robustness of coherence (ROC)** - Let \( \mathcal{D}(C^d) \) be the convex set of density operators acting on a \( d \)-dimensional Hilbert Space. Let \( I \subset \mathcal{D}(C^d) \) be the subset of incoherent states. Then, the robustness of coherence of a state \( \rho \in \mathcal{D}(C^d) \) is defined as:

\[
C_{\text{ROC}}(\rho) = \min_{\tau \in \mathcal{D}(C^d)} \left\{ s \geq 0 \mid \frac{\rho + s \tau}{1 + s} =: \delta \in I \right\}. \tag{1}
\]

Clearly \( C_{\text{ROC}}(\rho) \) is the minimum weight of another state \( \tau \) such that its convex mixture with \( \rho \) yields an incoherent state \( \delta \). It is slightly different from the similarly defined robustness of entanglement [32] in that the mixing is not only over free, i.e. incoherent states in this case.

The robustness of coherence has an operational interpretation as a coherence witness through a semidefinite program. It also means that \( C_{\text{ROC}}(\rho) \) can be evaluated via a semidefinite program that finds the optimal coherence witness operator. This semidefinite program has been used to carry out the numerical calculations in this paper.

### III. PRELIMINARY RESULTS

In this section we derive two results on quantum coherence for joint states.

**Result I:** For any pure state \( |\psi_{AB}\rangle \), ROC is super-additive.

**Proof.** For any pure state \( |\psi_{AB}\rangle \), we have \( C_{\text{ROC}}(|\psi_{AB}\rangle) = C_1(|\psi_{AB}\rangle) \). Now, we use the superadditivity of the \( l_1 \)-norm of coherence along with the fact that ROC is always upper bounded by the the \( l_1 \)-norm of coherence to obtain \( C_{\text{ROC}}(|\psi_{AB}\rangle) = C_1(|\psi_{AB}\rangle) \geq C_1(\rho_A) + C_1(\rho_B) \geq C_{\text{ROC}}(\rho_A) + C_{\text{ROC}}(\rho_B) \), thus proving the result.

We now show that adding an incoherent ancilla doesn’t change the amount of coherence in a system. This intuitively obvious statement is shown below to hold for arbitrary legitimate coherence measures. In order to prove this, we note that the inequality \[ C_{\text{ROC}}(\rho) \leq (1 + k)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | \] for any set of states \( \rho \) and any incoherent state \( \sigma_B \), is a legitimate coherence measure. In order to prove this, we note that the inequality \[ C_{\text{ROC}}(\rho) \leq (1 + k)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | \] for any set of states \( \rho \) and any incoherent state \( \sigma_B \), is a legitimate coherence measure.

**Result II:** For any state \( \rho_A \) and any incoherent state \( \sigma_B \), \( C(\rho_A \otimes \sigma_B) = C(\rho_A) \) for any legitimate coherence measure \( C \).

**Proof.** Let us assume that \( \text{dim}(H_A) = \text{dim}(H_B) = n \geq 2 \). Clearly, \( \rho_A \otimes \sigma_B \) is a \( n^2 \times n^2 \) sparse matrix with its sparsity \( 1 - \frac{1}{n} \geq \frac{1}{2} \). As the dimension \( n \) increases, the sparsity also increases. Let \( \chi = C_{\text{ROC}}(\rho_A \otimes \sigma_B) \). Given a sparse matrix, we can always use permutation matrices to transform it to a matrix in block-diagonal form [34] = \( d_1 \rho_A \otimes d_2 \rho_A \otimes \ldots \otimes d_n \rho_A \) via permutation matrices [35]. Now, from (2) [33], we have, for any legitimate coherence measure \( C \), \( C(\rho_A \otimes \sigma_B) = C(d_1 \rho_A \otimes d_2 \rho_A \otimes \ldots \otimes d_n \rho_A) = \sum_{i=1}^{n} d_i C(\rho_A) = C(\rho_A) \), where the last line follows from the unit trace condition for density matrices.

### IV. SUB-ADDITIONALITY OF ROBUSTNESS OF COHERENCE

In this section, we explore the possible sub-additivity of ROC. To this end, we introduce the following class of \( n \)-qubit states \( \rho_{A_1 A_2 A_3 \ldots A_n} = (1 + k)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | \) for any set of states \( \rho \) and any incoherent state \( \sigma_B \), is a legitimate coherence measure. In order to prove this, we note that the inequality \[ C_{\text{ROC}}(\rho_A \otimes \sigma_B) = C(\rho_A) \] for any legitimate coherence measure \( C \).

**Theorem 1:** For an arbitrary \( n \)-qubit system \( A_1 A_2 A_3 \ldots A_n \), the ROC for the family \( \Sigma \) of states \( \rho_{A_1 A_2 A_3 \ldots A_n} = (1 + k)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | \) for any \( 0 \leq k \leq \frac{1}{2^n} \) and \( |\psi_1\rangle = \frac{1}{\sqrt{\sum_{i=1}^{n}|i\rangle}} \) is the maximally coherent \( n \)-qubit state, satisfies the following sub-additive relation:

\[
C_{\text{ROC}}(\rho_{A_1 A_2 A_3 \ldots A_n}) \leq \sum_{i=1}^{n} C_{\text{ROC}}(\rho_{A_i}). \tag{3}
\]

**Proof.** Given, \( \rho_{A_1 A_2 A_3 \ldots A_n} = (1 + k)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | \) for any \( 0 \leq k \leq \frac{1}{2^n} \) and \( |\psi_1\rangle = \frac{1}{\sqrt{\sum_{i=1}^{n}|i\rangle}} \) is the maximally coherent \( n \)-qubit state. Now, by using definition of robustness of coherence (1), we prepare a convex mixture \( \chi \) of an arbitrary \( n \)-qubit state \( \tau \) and \( \rho_{A_1 A_2 A_3 \ldots A_n} \), that is, mathematically expressed as:

\[
\chi = \left(1 + k\right)^{\frac{1}{2^n}} - k|\psi_1\rangle\langle \psi_1 | + s^r \frac{1 + s}{1 + s}, \tag{4}
\]

where \( s \) is \( C_{\text{ROC}}(\rho_{A_1 A_2 A_3 \ldots A_n}) \). Without any loss of generality, when \( \chi \) in Eq.\,(4) is expanded in \( n \)-qubit computational basis, the diagonal elements are of the form

\[
\chi_{ii} = \frac{1 + 2^n s^r}{2^n (1 + s)} \tag{5}
\]
whereas, the off-diagonal elements are of the form
\[ \chi_{ij} = \frac{-k + 2^n s \tau_{ij}}{2^n (1 + s)}. \]  

(6)

For \( \chi \) in Eq.(4) to be an incoherent state, we have to ensure that the off-diagonal elements of \( \chi \), described by Eq.(6), will be zero. So, by equating Eq.(6) to zero, we finally arrive at the following condition:
\[ s = \frac{k}{2^n \tau_{ij}}. \]  

(7)

As per definition of robustness of coherence(Eq. (1)), \( s \in \mathbb{R} \), where \( \mathbb{R} \) is the set of Real numbers, has to be minimized. Since \( s \in \mathbb{R} \), so, clearly, \( \tau_{ij} \in \mathbb{R} \). Now, in the trivial case, \( s \) is zero when \( \rho_{A_1 A_2 A_3 \ldots A_n} \) is already an incoherent state. In the non-trivial case, \( s \) is minimum when \( \tau_{ij} \) takes the maximum value of \( k \), i.e., \( \tau_{ij} = \frac{1}{2^n - 1} \). Hence, after substituting \( \tau_{ij} = \frac{1}{2^n - 1} \) in Eq.(7), we have,
\[ s = C_{ROC}(\rho_{A_1 A_2 A_3 \ldots A_n}) = k \left( 1 - \frac{1}{2^n} \right). \]  

(8)

Now, let us consider the single-qubit subsystems
\[ \rho_{A_i} = \text{Tr}_{A_{i+1} A_{i+2} \ldots A_n}[\rho_{A_1 A_2 A_3 \ldots A_n}] = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \]  
in computational basis. For single qubit systems, we know that robustness of coherence is equal to its \( l_1 \)-norm of coherence for a fixed basis. Hence, for single qubit computational basis, \( C_{ROC}(\rho_{A_i}) = C_{l_1}(\rho_{A_i}) = k \).

Finally, we have,
\[ \Lambda = C_{ROC}(\rho_{A_1 A_2 A_3 \ldots A_n}) - \sum_{i=1}^{n} C_{ROC}(\rho_{A_i}) = k \left( 1 - \frac{1}{2^n} \right) - nk = k \left( 1 - \left(n + \frac{1}{2^n} \right) \right). \]  

(9)

Clearly, for \( n \in \mathbb{Z}^+ \) and \( 0 \leq k \leq \frac{1}{2^n} \), we have \( \Lambda \leq 0 \). Hence, proved. 

\[ \square \]

Given any pure state, its ROC is identical with its \( l_1 \)-norm of coherence, which is always super-additive. We now turn to the scenario when elements of the set of states \( \Sigma \) mentioned in the previous theorem are mixed with a given pure state \( |\phi\rangle \) and investigate what happens to the sub-additivity property as we increase the mixing.

To this end, we randomly pick a large number of states \( |\sigma\rangle \) from the family \( \Sigma \) and mix every such state with a chosen pure state \( |\phi\rangle \) with mixing parameter \( p \) to obtain a large number of states \( \Sigma_p = \{(1-p)|\sigma\rangle + p|\phi\rangle\langle \phi|\} \). We want to know the probability of any randomly chosen element of this set satisfying the sub-additivity condition Eq.(3). Clearly, if \( p = 0 \), this set is a random subset of \( \Sigma \), thus all the elements will satisfy the sub-additivity condition. In the opposite limit, if \( p = 1 \), this set consists of only \( |\phi\rangle \), i.e., always super-additive for ROC. However, it is the intermediate region which is of interest to us. For simplicity, we confine ourselves to the 2-qubit scenario. We consider two different pure states \( |\phi\rangle \), one being the maximally coherent state \( |\phi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \), the other being the maximally entangled state \( |\phi_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \). For each of them and every value of the mixing weight \( p \), choosing 10,000 random states from \( \Sigma_p \) according to Haar measure, we calculate the percentage of states in the set \( \Sigma_p \) which satisfy the subadditivity condition. FIG 1 captures the result. Two properties of this figure are quite interesting. Firstly, the plots are almost identical for two very different sets of pure states \( |\phi_i\rangle \)\( i = 1, 2 \), viz. the maximally coherent states(indicated by red points) and the maximally entangled states(indicated by blue points). Secondly, instead of the proportion of states satisfying subadditivity condition (3) diminishing smoothly as \( p \to 1 \), it shows a sudden death at around \( p = 0.25 \).

\[ \text{FIG. 1: (color online) Percentage of randomly chosen two qubit states from } \Sigma_p \text{ which satisfies subadditivity vs. the mixing weight } p, \text{ where the pure state } |\phi\rangle \text{ is either the two qubit maximally coherent state (red dots) or the two qubit maximally entangled state(blue dots). 1000 randomly generated states taken for each value of } p. \]

V. ORDERING OF STATES THROUGH DIFFERENT COHERENCE MEASURES

Quantification of any resource through some measure begs the question - what is the operational significance of that particular measure? Indeed the same resource can be operationally relevant in many different protocols. This naturally leads us to the next question: if the same resource
is quantified by different measures motivated by different protocols - then can a state which is bad for a particular protocol turn out to be good for another protocol utilizing the same resource?

For the resource theory of coherence - a central question is, when can one transform a quantum state $\rho$ to $\sigma$ using incoherent operations? If both input and target states are pure, say $|\psi\rangle$ and $|\chi\rangle$ respectively, a necessary and sufficient condition for such convertibility [36] is given by:

$$\vec{c}_\phi \prec \vec{c}_\chi,$$

where $\vec{c}_\xi$ for any state $|\xi\rangle$ is the collection of squared moduli of the coefficients of that state when expanded out in the basis of our choice. Evidently it is possible to have pairs of pure states for which the collection of coefficients do not majorize each other. This leaves open the possibility that even for pairs of such pure states, two different coherence measures may give us different ordering. This is indeed confirmed for pure as well as mixed states [37] for $C_{\text{rel}}$ and $C_1$. In this section, we investigate the statistics of ordering for different coherence measures, viz. $C_{\text{rel}}, C_1, \text{ and } C_{\text{ROC}}$. If random states are chosen from the state space according to Haar measure. We decided to check the percentage of randomly chosen pairs of states with different ordering wrt pairwise chosen coherence measures depending upon dimension and rank of the chosen states. [38]

From Figure 2, it is evident that as the dimension of the quantum state increases, the percentage of ordering violations between robustness of coherence and relative entropy measure of coherence (denoted by the green curve) remains greater than that of between robustness of coherence and $l_1$-norm of coherence (denoted by the blue curve) and $l_1$-norm and relative entropy of coherence (denoted by the red curve). Moreover, we observe that for dimension $d \leq 5$, the percentage of ordering violation between $l_1$-norm and relative entropy of coherence is greater than that of between robustness of coherence and $l_1$-norm of coherence. However, for dimensions $d > 5$, the percentage of ordering violation between $l_1$-norm and robustness of coherence is greater than that of between relative entropy of coherence and $l_1$-norm of coherence.

In Figure 3, we observe a similar trend as that of Figure 2. Here, as the rank of the quantum state increases, the percentage of ordering violations between robustness of coherence and relative entropy measure of coherence is significantly greater than that of between robustness of coherence and $l_1$-norm of coherence and $l_1$-norm and relative entropy of coherence. For pure states, i.e. states of rank 1, robustness of coherence is identical to the $l_1$ norm of coherence, therefore there is no ordering violation among them. However, for mixed states, the percentage of ordering violation between $l_1$-norm and robustness of coherence is greater than that of between relative entropy of coherence and $l_1$-norm of coherence.

VI. CONCLUSION

We conclude that unlike $l_1$-norm or relative entropy of coherence, which are super-additive, ROC can be sub-additive for certain classes of states. If we take a mixture of that class of states and pure states, we have found out that beyond a cer-
tain range of mixing weight, such mixtures cease to satisfy sub-additive property. We have found that for a pair of randomly generated density matrices, there exists a possibility of ordering violations corresponding to different legitimate measures of coherence. We welcome further work on implications of sub-additivity of ROC for quantum advantage in phase discrimination tasks and quantum information theory in general.

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[35] Since permutations correspond merely to relabeling of the basis vectors, amount of coherence of a system does not depend on such permutations.


[38] Why both dimension and rank ? The explanation is that for higher dimensional states, when we generate datasets of random quantum states, we miss out the states of lower ranks which are of measure zero.