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2D BEAMFORMING ON SPARSE ARRAYS WITH SPARSE BAYESIAN LEARNING

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ABSTRACT

Sparse arrays such as co-prime and nested arrays can identify more sources than the number of sensors. This is because their difference co-arrays contain a uniformly spaced virtual array with more elements than the number of sensors in the array. In this paper we demonstrate this using two dimensional co-prime and nested sparse arrays combined with sparse Bayesian learning (SBL) for 2D beamforming in azimuth and elevation. SBL can directly process the sparse array data and significantly outperform conventional beamforming and MUSIC as seen from simulations.

***Index Terms*—** 2D beamforming, SBL, co-prime arrays, nested arrays, compressive sensing

1. INTRODUCTION

A two-dimensional (2D) planar array can estimate the azimuth and elevation directions (2D beamforming) of plane waves incident on it. The uniform rectangular array (URA) often used for this purpose is too dense and requires large number of sensors to achieve a given resolution. Sparse arrays can significantly reduce the number of sensors and resolve more source directions than the number of sensors when the incident signals are spatially wide sense stationary.

Various sparse array geometries have been proposed in the literature, here we focus on co-prime and nested arrays. In one-dimension, linear co-prime [1] and nested [2] arrays are well studied [3, 4, 5, 6, 7, 8]. More recently, 2D nested arrays [9, 10] and multi-dimensional co-prime arrays [11] have been proposed and used for 2D direction-of-arrival (DOA) estimation [12, 13, 14]. In this paper we use the terms beamforming and DOA estimation synonymously.

Sparse Bayesian learning (SBL) [15, 16] is a sparse processing method that has been successfully used for beamforming with uniform linear arrays [17, 18, 19, 20, 21] and 1D sparse arrays [7, 8, 22]. SBL formulates the beamforming problem probabilistically and performs stochastic maximum likelihood to estimate the source covariance parameter. Sparsity profile of the source covariance diagonal corresponds to the source direction-of-arrivals (DOAs). Our previous work [7, 8] explored SBL for processing measurements from 1D sparse arrays. In this paper we extend our analysis for 2D sparse arrays. We give examples of 2D sparse arrays and use them to beamform source azimuth and elevation parameters. The sparse arrays also identify more sources than the number of sensors.

2. 2D SPARSE ARRAYS

In this paper we are concerned with 2D planar arrays. A 2D uniform rectangular array (URA) has sensors located on a uniform grid with spacing $d = \lambda/2$ along both directions. Here d is the fundamental grid spacing. For a given aperture a URA has a dense spacing of sensors to avoid aliasing.

Sparse arrays are an alternative to URA which use relatively fewer sensors. When the incident signals are wide sense stationary, sparse arrays can capture all the degrees of freedom captured by the corresponding full URA. There are many classes of sparse arrays, two of which are used in this paper, co-prime and nested arrays, and are briefly discussed next. A detailed mathematical analysis of the 2D co-prime [11] and nested [9, 10] arrays is beyond the scope of this paper.

2.1. Co-prime arrays

2D co-prime arrays are constructed by combining two URAs each of which are generated using a 2×2 sampling matrix. These sampling matrices are relatively co-prime and their product commutes (see [11] for definitions and details). An example of a 2D co-prime array is shown in Fig. 1(a) and its difference co-array in Fig. 1(c). The array consists of a 9-sensor (red dots) URA with $2 \times \lambda/2$ spacing in both directions and a 16-sensor (blue dots) URA with spacing $3 \times \lambda/2$. The difference co-array (or simply co-array) is obtained by computing the pairwise differences between all the sensor positions. The number of elements (239) in the co-array is indicative of the degrees of freedom resolvable by the array.

2.2. Nested arrays

A two dimensional nested array consists of two URAs: a dense URA and a sparse URA. A 2D nested array example is shown in Fig. 1(b) along with its difference co-array in Fig. 1(d). The array consists of a dense 9-sensor (red dots) URA with $1 \times \lambda/2$ spacing in both directions and a less dense

16-sensor (blue dots) URA with spacing $3 \times \lambda/2$. The difference co-array of a nested array has no holes while that of a co-prime array has holes which is seen in Fig. 1. The number of elements (263) in the nested-array is indicative of the degrees of freedom resolvable by the array, which is similar to that of the co-prime array. Though the examples of sparse arrays considered here lie on rectangular grids, the analysis we develop is valid for other array realizations as well.

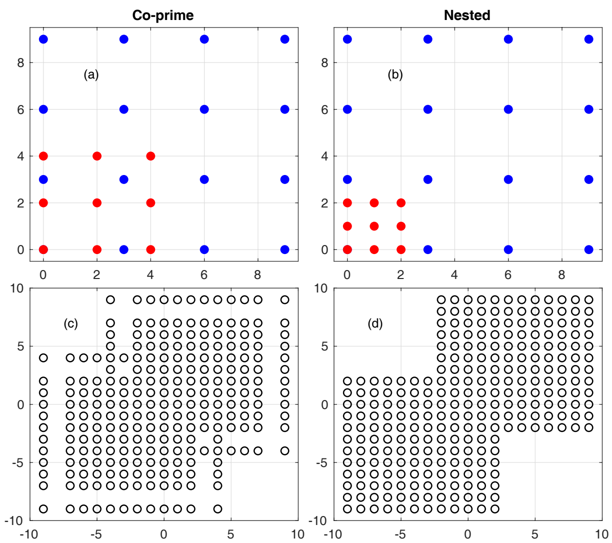

Fig. 1. 2D Sparse arrays: Examples of (a) Co-prime and (b) Nested arrays along with their difference coarrays in (c) and (d) respectively. For the physical arrays in (a) and (b) the two colors correspond to the two subarrays which make up the sparse array.

3. 2D BEAMFORMING WITH SBL

Availability of two dimensional arrays allows DOA estimation in azimuth and elevation simultaneously. Let the location of the N sensors be $\{\mathbf{d}_1, \dots, \mathbf{d}_N\}$. The kth plane wave incident on this array with elevation θ_k and azimuth ϕ_k has the following relative phase at the nth sensor:

$$[\mathbf{a}(\theta_k, \phi_k)]_n = e^{-j\frac{2\pi}{\lambda}(d_n^x \cos\phi_k \sin\theta_k + d_n^y \sin\phi_k \sin\theta_k)}, \quad (1)$$

where $\mathbf{d}_n = [d_n^x, d_n^y]^T$. The vector $\mathbf{a}(\theta_k, \phi_k)$ is called the steering vector. The noisy sensor observations are given by

$$\mathbf{y} = \sum_{k=1}^{K} s_k \mathbf{a}(\theta_k, \phi_k) + \mathbf{n}, \quad (2)$$

where s_k is the complex amplitude of the kth source, K is the number of sources present, and $\mathbf{n}$ is the noise. The 2D beamforming problem is to estimate source directions (θ_k, ϕ_k) and amplitudes s_k from the multi-snapshot observations $\mathbf{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_L]$. Though highly non-linear, following a similar approach as in 1D beamforming [23, 18], this can be cast as an underdetermined system of linear equations.

SBL Algorithm
1. Input: $\mathbf{Y}, \mathbf{A}, \sigma^2$
2. Parameters: $\epsilon = 10^{-3}, N_t = 500$
3. Initialization: $\gamma_m^{\text{old}} = \mathbf{a}_m^H \mathbf{S}_\mathbf{y} \mathbf{a}_m, \forall m$
4. **for** $i = 1$ to N_t
5. Compute: $\mathbf{\Sigma}_\mathbf{y} = \sigma^2 \mathbf{I}_N + \mathbf{A}\mathbf{\Gamma}^{\text{old}}\mathbf{A}^H$
6. γ_m^{new} update $\forall m$ using (8)
7. If $\frac{\lVert\gamma^{\text{new}} - \gamma^{\text{old}}\rVert_1}{\lVert\gamma^{\text{old}}\rVert_1} < \epsilon$, **break**
8. $\gamma^{\text{old}} = \gamma^{\text{new}}$, $\mathbf{\Gamma}^{\text{old}} = \text{diag}(\gamma^{\text{new}})$
9. **end**
10. Output: γ

Table 1. SBL algorithm pseudocode: Input consists of data matrix $\mathbf{Y}$, dictionary $\mathbf{A}$, and noise variance σ^2. Convergence is controlled by the error threshold ϵ and maximum number of iterations N_t.

Let $(\theta_1, \phi_1), \dots, (\theta_M, \phi_M)$ be a discrete grid in the $\theta - \phi$ plane. Assume that this grid is sufficiently fine so that the true source directions are among these M candidate directions. Let $\mathbf{A} = [\mathbf{a}_1, \dots, \mathbf{a}_M]$ be the dictionary of steering vectors with $\mathbf{a}_i$ corresponding to direction (θ_i, ϕ_i). The observation $\mathbf{y}$ is expressed as

$$\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{n}, \quad (3)$$

where $\mathbf{x} \in \mathbb{C}^M$ has exactly K non-zero entries at locations corresponding to the K source directions and having value of source amplitudes. Typically $K \ll M$ which makes (3) an undetermined system which can be solved using compressive sensing methods. In this paper we use SBL to solve for the covariance of the complex source amplitudes $\mathbf{x}$ in (3).

We briefly discuss the Sparse Bayesian learning algorithm. For more detailed description readers can refer to any of the numerous references [15, 16, 18, 21]. When multiple snapshots are available, (3) can be written as

$$\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{N}, \quad (4)$$

where the noise $\mathbf{N} = [\mathbf{n}_1, \dots, \mathbf{n}_L]$ is zero-mean complex Gaussian with variance σ^2, $\mathbf{n}_l \sim \mathcal{CN}(\mathbf{n}_l; \mathbf{0}, \sigma^2\mathbf{I})$; $\mathbf{X} = [\mathbf{x}_1, \dots, \mathbf{x}_L]$ is the matrix of sparse source amplitudes. Assuming the sources are stationary over time, all the columns in $\mathbf{X}$ share the same sparsity profile. The observations are assumed to be independent across snapshots giving the multi-snapshot likelihood function

$$p(\mathbf{Y}|\mathbf{X}) = \prod_{l=1}^{L} p(\mathbf{y}_l|\mathbf{x}_l) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{y}_l; \mathbf{A}\mathbf{x}_l, \sigma^2\mathbf{I}). \quad (5)$$

In SBL, the source amplitudes are treated as zero-mean complex Gaussian random vectors with diagonal covariance $\mathbf{\Gamma} =$

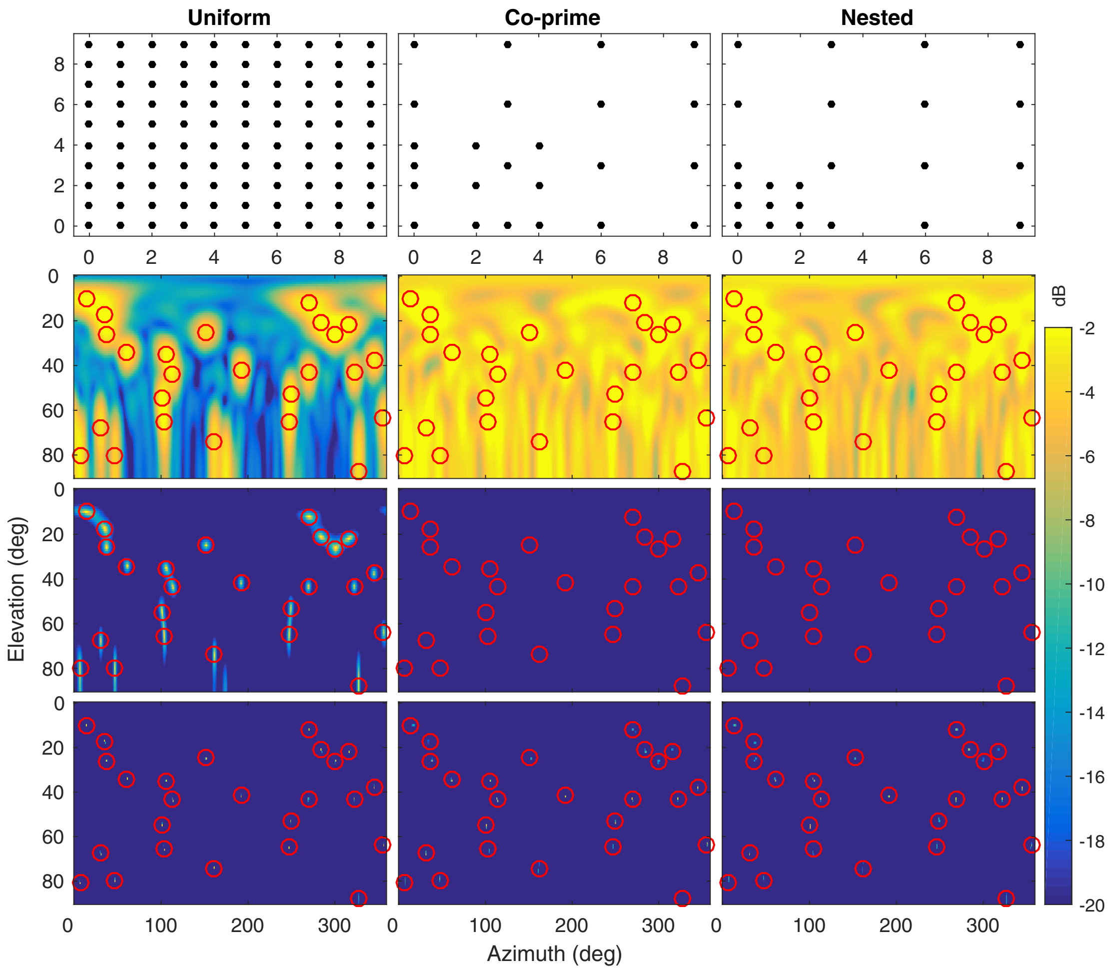

Fig. 2. Sensor positions for various 2D planar arrays (top row). Estimated 2D spectrum when $K = 25$ sources are present using CBF (second row), MUSIC (third row), and SBL (last row). Circle '$\circ$' indicates true source positions.

$\text{diag}(\gamma_1 \dots \gamma_M) = \text{diag}(\boldsymbol{\gamma})$. The prior distribution is given by

$$p(\mathbf{X}) = \prod_{l=1}^{L} p(\mathbf{x}_l) = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{x}_l; \mathbf{0}, \mathbf{\Gamma}). \quad (6)$$

From the Gaussian prior (6) and likelihood (5), the evidence term $p(\mathbf{Y})$ is also Gaussian and given by

$$p(\mathbf{Y}) = \int p(\mathbf{X}) p(\mathbf{Y}|\mathbf{X}) d\mathbf{X} = \prod_{l=1}^{L} \mathcal{CN}(\mathbf{y}_l; \mathbf{0}, \mathbf{\Sigma}_\mathbf{y}), \quad (7)$$

where $\mathbf{\Sigma}_\mathbf{y} = \sigma^2 \mathbf{I} + \mathbf{A}\mathbf{\Gamma}\mathbf{A}^H$. In SBL we estimate the diagonal entries of $\mathbf{\Gamma}$ by maximizing the (log) evidence i.e. log $p(\mathbf{Y})$

$$(\hat{\gamma}_1 \dots \hat{\gamma}_M) = \arg\max_{\boldsymbol{\gamma}} \left\{ -\sum_{l=1}^{L} \left(\mathbf{y}_l^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{y}_l + \log|\mathbf{\Sigma}_\mathbf{y}| \right) \right\}.$$

Differentiating the above objective function and equating the derivatives to zero gives the fixed point update rule (for details see [15, 16, 18, 21])

$$\gamma_m^{\text{new}} = \gamma_m^{\text{old}} \frac{\text{Tr}[\mathbf{S}_\mathbf{y} \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m \mathbf{a}_m^H \mathbf{\Sigma}_\mathbf{y}^{-1}]}{\mathbf{a}_m^H \mathbf{\Sigma}_\mathbf{y}^{-1} \mathbf{a}_m}, \quad (8)$$

where $\mathbf{S}_\mathbf{y} = \frac{1}{L}\mathbf{Y}\mathbf{Y}^H$ is the sample covariance matrix (SCM) and Tr$[\cdot]$ the trace operator. The SBL pseudocode is given in Table 1. The noise variance is also required to be estimated [17, 18], here we assume noise to be known. The parameters ϵ and N_t are the convergence error threshold and number of iterations of the algorithm. The unknown vector $\boldsymbol{\gamma}$ is initialized to the conventional beamformer (CBF) output.

As discussed in [15], estimate of $\boldsymbol{\gamma}$ by SBL is sparse. Since the element γ_m of $\boldsymbol{\gamma}$ corresponds to the variance of x_m, when $\gamma_m = 0$ it implies that $x_m = 0$. Hence a sparse $\boldsymbol{\gamma}$ corresponds to sparse source amplitude vector $\mathbf{x}$. The SBL algorithm in Table 1 when applied to sparse array observations can identify more sources than the number of sensors [7, 8].

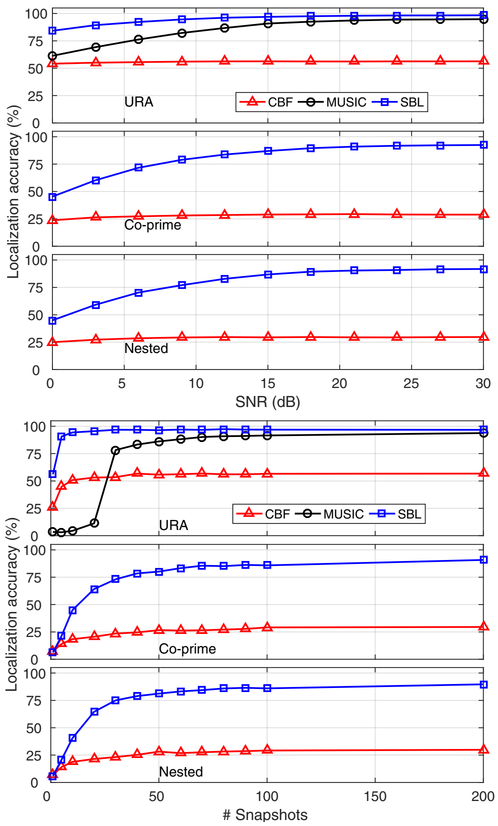

Fig. 3. 2D Beamforming: Percentage of sources localized over 100 Monte Carlo simulations for changing SNR (top) and changing snapshot number (bottom). All the arrays have same aperture, number of sources is $K = 25$.

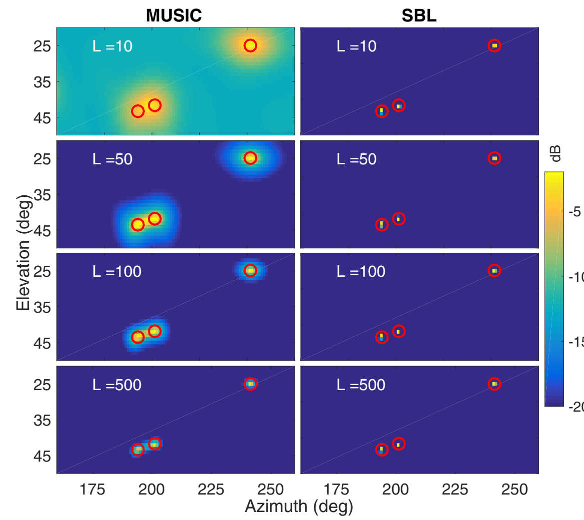

Fig. 4. 2D Beamforming: 2D Spectrum for MUSIC (left) and SBL (right) as the number of snapshots is increased as $L = 10, 50, 100, \&500$. The array is a URA with 100 sensors.

4. SIMULATIONS

We compare 2D beamforming results of SBL with conventional beamforming (CBF) [24] and MUSIC [25, 26]. Three arrays with similar aperture are compared (URA, co-prime, and nested) as shown in Fig. 2 (top row) with basic array spacing of $\lambda/2$. The URA consists of 10$\times$10=100 sensors while co-prime and nested arrays have 9+16-1=24 sensors each. We look for sources every 1^o in elevation $[0, 90]^o$ and azimuth $[0, 360]^o$, giving $91 \times 360 = 32760$ possible source locations.

$K = 25$ sources are generated with uniformly random azimuth and elevation directions and each with source variance $\gamma = 1$. A total of $L = 100$ snapshots each with an array SNR of 20 dB are simulated. The normalized spectrum obtained by CBF (second row), MUSIC (third row), and SBL (last row) are shown in Fig. 2 for a random data realization. All three methods work directly on the observed SCM $\mathbf{S_y}$.

For the URA, CBF can distinguish between relatively distant sources but often clusters nearby sources together. MUSIC provides super-resolution with the URA as there are 25 sources and 100 sensors. For co-prime and nested arrays the CBF spectrum is non-informative and the MUSIC spectrum does not exist as the number of sensors (24) is less than the number of sources (25). SBL provides best resolution for all the three arrays (URA, co-prime, and nested). It can accurately perform localization even when number of sources exceeds the number of sensors.

For various algorithms, percentage of the total sources localized (averaged over 50 Monte Carlo simulations) for $K = 25$ sources is shown in Fig. 3. A source is said to be correctly localized if there is a peak in the spectrum within 3^o radius of true source location. The localization accuracy is plotted as a function of SNR (Fig. 3, top) and as a function of number of snapshots (Fig. 3, bottom). SBL performs significantly better than CBF for all arrays. SBL performs better than MUSIC for the URA especially for low SNR and low snapshot scenarios. The superior resolution of SBL when compared to MUSIC is also seen when two DOAs are nearby and increasingly more snapshots are used for processing (Fig. 4). In fact for the URA just 10 snapshots with SNR of 15 dB give accurate localization with SBL while MUSIC performs poorly because of snapshot deficiency. In future work we will include experimental data results and comparison with co-array MUSIC [3] which can identify more sources than sensors.

5. CONCLUSIONS

We provided an introduction to 2D co-prime and nested arrays for 2D beamforming. SBL is advantageous since it can directly work with sparse array data. As observed in simulations, compared to CBF and MUSIC, SBL has higher accuracy of DOA localization while using fewer snapshots and in low SNR scenarios. Also, SBL combined with sparse arrays can identify more sources than the number of sensors.

6. REFERENCES

[1] P. P. Vaidyanathan and P. Pal, "Sparse sensing with co-prime samplers and arrays," *IEEE Trans. Signal Process.*, vol. 59, no. 2, pp. 573–586, Feb. 2011.

[2] P. Pal and P. P. Vaidyanathan, "Nested arrays: A novel approach to array processing with enhanced degrees of freedom," *IEEE Trans. Signal Process.*, vol. 58, no. 8, pp. 4167–4181, Aug. 2010.

[3] P. Pal and P. P. Vaidyanathan, "Coprime sampling and the MUSIC algorithm," *Proc. IEEE Digital Signal Process. Workshop IEEE Signal Process. Edu. Workshop*, Jan. 2011.

[4] K. Adhikari, J. R. Buck, and K. E. Wage, "Extending coprime sensor arrays to achieve the peak side lobe height of a full uniform linear array," *EURASIP J. Advances in Signal Process.*, vol. 2014, no. 148, Sep. 2014.

[5] D. Bush and N. Xiang, "Broadband implementation of coprime linear microphone arrays for direction of arrival estimation," *J. Acoust. Soc. Am.*, vol. 138, no. 1, pp. 447–456, 2015.

[6] S. Qin, Y. D. Zhang, and M. G. Amin, "Generalized coprime array configurations for direction-of-arrival estimation," *IEEE Trans. Signal Process.*, vol. 63, no. 6, pp. 1377–1390, Mar. 2015.

[7] S. Nannuru, P. Gerstoft, A. Koochakzadeh, and P. Pal, "Sparse Bayesian learning for DOA estimation using co-prime and nested arrays," in *IEEE Sensor Array Multichannel Signal Process. Workshop*, Sheffield, U.K., Jul. 2018.

[8] S. Nannuru, A. Koochakzadeh, K. L. Gemba, P. Pal, and P. Gerstoft, "Sparse Bayesian learning for beamforming using sparse linear arrays," *J. Acoust. Soc. Am.*, vol. 144, no. 5, pp. 2719–2729, Oct. 2018.

[9] P. Pal and P. P. Vaidyanathan, "Nested arrays in two dimensions, part I: Geometrical considerations," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4694–4705, Sep. 2012.

[10] P. Pal and P. P. Vaidyanathan, "Nested arrays in two dimensions, part II: Application in two dimensional array processing," *IEEE Trans. Signal Process.*, vol. 60, no. 9, pp. 4706–4718, Sep. 2012.

[11] P. P. Vaidyanathan and P. Pal, "Theory of sparse coprime sensing in multiple dimensions," *IEEE Trans. Signal Process.*, vol. 59, no. 8, pp. 3592–3608, Aug. 2011.

[12] X. Lan, Y. Li, and E. Wang, "A RARE algorithm for 2D DOA estimation based on nested array in massive MIMO system," *IEEE Access*, vol. 4, pp. 3806–3814, Jun. 2016.

[13] D. Zhang, Y. Zhang, G. Zheng, B. Deng, C. Feng, and J. Tang, "Two-dimensional direction of arrival estimation for coprime planar arrays via polynomial root finding technique," *IEEE Access*, vol. 6, pp. 19540–19549, Apr. 2018.

[14] C. Li, L. Gan, and C. Ling, "2D MIMO radar with coprime arrays," in *IEEE Sensor Array Multichannel Sig. Process. Workshop*, Sheffield, U.K., Jul. 2018.

[15] M. E. Tipping, "Sparse Bayesian learning and the relevance vector machine," *J. Mach. Learn. Res.*, vol. 1, pp. 211–244, Jun. 2001.

[16] D. P. Wipf and B. D. Rao, "An empirical Bayesian strategy for solving the simultaneous sparse approximation problem," *IEEE Trans. Signal Process.*, vol. 55, no. 7, pp. 3704–3716, Jul. 2007.

[17] Z.-M. Liu, Z.-T. Huang, and Y.-Y. Zhou, "An efficient maximum likelihood method for direction-of-arrival estimation via sparse Bayesian learning," *IEEE Trans. Wireless Comm.*, vol. 11, no. 10, pp. 1–11, 2012.

[18] P. Gerstoft, C. F. Mecklenbräuker, A. Xenaki, and S. Nannuru, "Multi snapshot sparse Bayesian learning for DOA," *IEEE Signal Process. Lett.*, vol. 23, no. 10, pp. 1469–1473, Oct. 2016.

[19] K. L. Gemba, S. Nannuru, P. Gerstoft, and W. S. Hodgkiss, "Multi-frequency sparse Bayesian learning for robust matched field processing," *J. Acoust. Soc. Am.*, vol. 141, no. 5, pp. 3411–3420, May 2017.

[20] P. Gerstoft, C. F. Mecklenbräuker, W. Seong, and M. Bianco, "Introduction to compressive sensing in acoustics," *J. Acoust. Soc. Am.*, vol. 143, no. 6, pp. 3731–3736, Jun. 2018.

[21] S. Nannuru, K.L. Gemba, P. Gerstoft, W.S. Hodgkiss, and C. F. Mecklenbräuker, "Sparse Bayesian learning with multiple dictionaries," *Signal Process.*, DOI: 10.1016/j.sigpro.2019.02.003 2019.

[22] Y. Qin, Y. Liu, J. Liu, and Z. Yu, "Underdetermined wideband DOA estimation for off-grid sources with coprime array using sparse Bayesian learning," *Sensors*, vol. 18, no. 1, 2018.

[23] D. Malioutov, M. Çetin, and A. S. Willsky, "A sparse signal reconstruction perspective for source localization with sensor arrays," *IEEE Trans. Signal Process.*, vol. 53, no. 8, pp. 3010–3022, 2005.

[24] H. L. Van Trees, *Optimum Array Processing (Detection, Estimation, and Modulation Theory, Part IV)*, Wiley-Interscience, 2002.

[25] R. Schmidt, "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antenn. Prop.*, vol. 34, no. 3, pp. 276–280, Mar. 1986.

[26] P. P. Vaidyanathan and P. Pal, "Why does direct-MUSIC on sparse-arrays work?," in *Asilomar Conf. Signal, Systems Comp.*, Pacific Grove, CA, U.S.A., Nov. 2013.