

Dynamics of Structured Complex Recurrent Hopfield Networks

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Dynamics of Structured Complex Recurrent Hopfield Networks

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Abstract—In this research paper, Complex Recurrent Hopfield Networks are introduced. Dynamics of a Structured Complex Recurrent Hopfield Network whose synaptic weight matrix is skew Hermitian is studied. It is proved that such a network when operated in parallel mode leads to a cycle of length 4. Two new types of complex matrices known as Braided Hermitian and Braided Skew Hermitian are defined and dynamics of a complex recurrent Hopfield network based on them is studied. It is reasoned that a cycle of length 8 is reached in parallel mode of operation. Experimental results for different types of Complex Recurrent Hopfield Networks are also discussed.

Index Terms—Structured Complex Recurrent Hopfield Network, Associative Memory, Skew Hermitian Matrix

I. INTRODUCTION

The development of science and technology depended heavily on mathematical modeling of natural as well as artificial systems evolving in time. One of the most interesting systems is the biological neural network. McCulloch and Pitts proposed an interesting model of artificial neuron leading to extensive research on Artificial Neural Networks. In such networks, the inputs, synaptic weights, thresholds, outputs associated with a neuron are assumed to be real numbers. N.N Aizenberg removed this assumption and allowed those quantities to be complex numbers leading to the field of Complex Valued Neural Networks (CVNNs) [9].

Hopfield proposed a model of associative memory based on interconnection of McCulloch-Pitts neurons, driven by an initial state vector lying on real unit hypercube [8]. In a Hopfield neural network, the synaptic weight matrix is a symmetric matrix. Thus such a neural network constitute a recurrent neural network in which all connections are symmetric. In an effort to include directed cycles in the connectionist structure, the authors proposed a neural network whose synaptic weight matrix is asymmetric. Investigation of the dynamics of such network is reported in [4]. By restricting the weight matrix to be anti-symmetric, Goles et.al showed that a cycle of length 4 is always reached in the fully parallel mode of operation [7]. Such a network constitutes structured recurrent hopfield network.

In the research literature, there were several efforts related to Complex Hopfield neural network. The authors proposed certain interesting complex Hopfield network in [5] and studied the dynamics when the synaptic weight is Hermitian. A natural question that remained is the nature of the network dynamics when the synaptic weight matrix is Skew-Hermitian. This research paper is based on investigating the dynamics of such a structured complex recurrent Hopfield network. Investigation is also carried on studying the dynamics of special complex recurrent Hopfield networks where the synaptic matrices are Braided Hermitian and Braided Skew Hermitian matrices.

This research paper is organized as follows: In section II, Complex Recurrent Hopfield Network is introduced. In section III, the dynamics of the structured Complex Recurrent Hopfield Network is discussed. In section IV, the dynamics of some special complex hopfield networks is discussed. Experimental results for different types of Complex Recurrent Hopfield Network are shown in section V and VI. Applications of the proposed Structured Complex Recurrent Hopfield Network are discussed in section VII. Future Work is discussed in section VIII. The paper concludes in section IX.

II. COMPLEX RECURRENT HOPFIELD NETWORK

A Complex Recurrent Hopfield network is an Artificial Neural Network similar to a real Hopfield network with synaptic weights and the state of the neurons being complex values. It is a nonlinear dynamical system represented by a weighted, directed graph. The nodes of the graph represent artificial neurons and the edge weights correspond to synaptic weights. Each node/neuron contains a threshold value. Thus, a Complex Recurrent Hopfield neural network can be represented by a threshold vector T and a synaptic weight matrix M that is not necessarily Hermitian. Each neuron assumes a state which lies in $\{+1 + j, -1 - j, -1 + j, +1 - j\}$. The order of the network corresponds to the number of neurons. Such a network potentially has some directed cycles in the graph. Let the state of the i^{th} neuron at time 't' be denoted by

$$V_i(t) \text{ where } V_i(t) \in \{+1 + j, -1 - j, -1 + j, +1 - j\}.$$

Thus the state of the nonlinear dynamical system having ‘N’ neurons is represented by Nx1 complex vector, V(t). The state updation at the ith node is governed by the following equation,

$$V_i(t+1) = C \text{signum} \left(\sum_{j=1}^N M_{ij} V_j(t) - T_i \right) \quad (1)$$

where Csignum is the complex signum function defined as,

$$C \text{signum}(a + ib) = \begin{cases} 1 + j & \text{if } a \geq 0 \text{ and } b \geq 0 \\ 1 - j & \text{if } a \geq 0 \text{ and } b \leq 0 \\ -1 - j & \text{if } a \leq 0 \text{ and } b \leq 0 \\ -1 + j & \text{if } a \leq 0 \text{ and } b \geq 0 \end{cases}$$

Depending on the number of nodes at which the state update is taking place, the Complex Recurrent Hopfield Network operation can be classified into the following modes

- **Serial mode:** The state update in Eq.1 is performed exactly at one of the nodes/neurons at time t
- **Fully Parallel mode:** The state update in Eq.1 is performed simultaneously at all nodes/neurons at time t.

Before studying the dynamics of the Complex Recurrent Hopfield network, we need to define a suitable Energy function to the network depending on the mode of operation [1].

Similar to the energy functions defined in the real Hopfield network [2], we need to define different energy functions for both serial and fully parallel modes of operation.

Energy function for a network operating in Serial mode: The energy function for the network operating in a serial mode at a time ‘t’ can be defined as

$$E_1(t) = V(t)^+ * M * V(t)$$

Where V(t)⁺ denotes the conjugate transpose of V(t) and ‘*’ denotes matrix multiplication. The same notation will be followed throughout the paper.

Energy function for a network operating in fully parallel mode: The energy function for a network operating in fully parallel mode at a time ‘t’ requires the use of the state of the network at time ‘t-1’. The energy function is defined as

$$E_2(t) = \text{realpart}\{V(t-1)^+ * M * V(t)\}$$

To study the dynamics of the networks defined in this paper, we make use of the above defined energy functions. The structure of M decides the nature of the network. Here we consider two structures to M.

A. Complex Ordinary Hopfield Network

A Complex Ordinary Hopfield network has a Hermitian matrix as the synaptic weight matrix i.e. $M_{ij} = \overline{M_{ji}}$ for all i and j and $\overline{M_{ji}}$ denotes conjugate of element M_{ji} .

Theorem 1. Let the pair $R = (M, T)$ specify a Complex Ordinary Hopfield Network, (with M being a Hermitian Matrix) then the following hold true:

- If R is operating in a serial mode and the elements of the diagonal of M are non-negative, the network will always converge to a stable state (i.e. there are no cycles in the state space).
- If R is operating in the fully parallel mode, the network will always converge to a stable state or to a state cycle with cycle length 2 (i.e. the cycles in the state space are of length ≤ 2).

The proof of the above theorem can be found in [6] which uses the energy function $E_1(t)$ for the serial mode and the graph properties to prove the fully parallel mode.

The complex ordinary Hopfield network has the ability to store more information (data) than the ordinary Hopfield network and hence is more beneficial to use.

B. Structured Complex Recurrent Hopfield Network:

Inspired from the fact that when a real ordinary Hopfield network having the synaptic weight matrix M as an Anti-Symmetric matrix operated in a fully parallel mode, a state cycle of length 4 is always observed [7], we are motivated to study the case for a Complex Recurrent Hopfield Network having the synaptic weight matrix M as a Skew-Hermitian matrix i.e. $M_{ij} = -\overline{M_{ji}}$. From now, such a network will be referred as Structured Complex Recurrent Hopfield network throughout this paper. The following section studies the dynamics of the Structured Complex Recurrent Hopfield Network.

III. DYNAMICS OF STRUCTURED COMPLEX HOPFIELD NETWORK

In this section we study the dynamics of the Structured Complex Recurrent Hopfield Network.

Theorem 2. Let $R = (M, T)$ specify a complex Recurrent Hopfield Network (with M being a Skew Hermitian Matrix). Then if the network is operated in a fully parallel way, a cycle of length 4 in the state space is observed.

Before discussing the proof of the above theorem, the following assumptions have to be made:

- The value of any element in the vector $M * V(t)$ is never zero where each element of V(t) lies in $\{1 + j, -1 - j, -1 + j, 1 - j\}$

Proof. In this proof, we will make use of the energy function defined for the fully parallel mode.

$$E_2(t) = \text{realpart}\{V(t-1)^+ * M * V(t)\}$$

We can see that the above can be equivalently written as

$$E_2(t) = V(t-1)^+ * M * V(t) + \text{conj}\{V(t-1)^+ * M * V(t)\}$$

where conj denotes Conjugate.

Consider change in energy between in the network between the times 't' and 't+1',

$$\begin{aligned} E_2(t+1) - E_2(t) &= V(t)^+ * M * V(t+1) - \\ &V(t-1)^+ * M * V(t) + \text{conj}\{V(t)^+ * M * V(t+1)\} - \\ &\text{conj}\{V(t-1)^+ * M * V(t)\} \end{aligned} \quad (2)$$

Here we need to note that since M is Skew Hermitian,

$$V(t-1)^+ * M * V(t) = -V(t)^T * \overline{M} * \overline{V(t-1)}$$

Let $V(t)^+ * M = A + B$ where A is real of $V(t)^+ * M$ and B is imaginary of $V(t)^+ * M$. Now, by replacing this in eq1, it leads to

$$\begin{aligned} E_2(t+1) - E_2(t) &= A * (V(t+1) + \overline{V(t-1)}) + \\ &B * (V(t+1) - \overline{V(t-1)}) + \text{conj}\{A * (V(t+1) + \overline{V(t-1)}) \\ &+ B * (V(t+1) - \overline{V(t-1)})\} \end{aligned}$$

which can be further simplified as,

$$\begin{aligned} E_2(t+1) - E_2(t) &= 2 * A * (V_R(t+1) + V_R(t-1)) + \\ &2 * B * (V_I(t+1) + V_I(t-1)) \end{aligned}$$

where $V_R(t)$ denotes real part of $V(t)$ and $V_I(t)$ denotes imaginary part of $V(t)$.

Here it should be noted that the sign of $V_R(t+1)_i$ and that of A_i is always same for all values of 'i' since $A = \text{realpart}\{V(t)^+ * M\}$ and $V(t+1) = \text{Csignum}(M * V(t))$ hence the value of $A * (V_R(t+1) + V_R(t-1))$ is always positive or zero.

Similarly the value of $B * (V_I(t+1) + V_I(t-1))$ is also always positive or zero. If the network has a loop initiated, then the change in energy of such a network should always be 0.

From the above equation, if we assume that A_i and B_i are non zero for all 'i', the networks change in energy will be zero only if

$$V(t+1) = -V(t-1)$$

i.e. the change in energy will be zero only if there is a cycle of length 4. \square

IV. DYNAMICS OF COMPLEX RECURRENT HOPFIELD NETWORKS

In this section we study the dynamics of two Special cases of Complex Recurrent Hopfield Networks for fully parallel mode of operation.

Braided Hermitian: A complex square matrix M is said to be Braided Hermitian matrix if it satisfies the condition $M = A + jA^T$ where A is a real matrix. Example is

$$\begin{bmatrix} 7 + 7j & 9 + 4j & -6 + 10j & 9 + 10j \\ 4 + 9j & -7 - 7j & -3 + 10j & 2 + 1j \\ 10 - 6j & 10 - 3j & -6 - 6j & 10 + 7j \\ 10 + 9j & 1 + 2j & 7 + 10j & -6 - 6j \end{bmatrix}$$

Braided Skew Hermitian: A complex square matrix M is said to be Braided Skew Hermitian matrix if it satisfies the condition, $M = A - jA^T$ where A is a real matrix. Example is

$$\begin{bmatrix} 5 - 5j & 6 - 4j & -3 - 10j & 4 - 6j \\ 4 - 6j & -5 + 5j & -6 + 2j & 1 + 4j \\ 10 + 3j & -2 + 6j & 3 - 3j & -4 - j \\ 6 - 4j & -4 - j & 1 + 4j & 5 - 5j \end{bmatrix}$$

A. Case 1 : M is Braided Hermitian matrix

So, let $M = A + jA^T$ where A is a real matrix. Now it leads to $M^T = j\overline{M}$ where \overline{M} is conjugate of M.

In this analysis, we will make use of the energy function defined for the fully parallel mode.

$$E_2(t) = \text{realpart}\{V(t-1)^+ * M * V(t)\}$$

Consider the change in energy,

$$E_2(t+1) - E_1(t) = \text{realpart}\{V(t)^+ * M * V(t+1) - V(t-1)^+ * M * V(t)\}$$

Let $V(t)^+ * M = C + jD$ where C and D are real matrices.

$$\begin{aligned} \Rightarrow V(t)^T * \overline{M} &= C - jD \\ \Rightarrow V(t-1)^+ * M * V(t) &= V(t)^T * M^T * \overline{V(t-1)} \\ \Rightarrow V(t-1)^+ * M * V(t) &= (C - jD) * j * \overline{V(t-1)} \end{aligned}$$

By substituting $V(t+1) = V_R(t+1) + jV_I(t+1)$ and $V(t-1) = V_R(t-1) + jV_I(t-1)$, where V_R and V_I denote real and imaginary parts of V, we get

$$E_2(t+1) - E_2(t) = C * (V_R(t+1) - V_I(t-1)) - D * (V_I(t+1) + V_R(t-1))$$

The change in energy, $E_2(t+1) - E_2(t)$ will be zero when a cycle is initiated. As we can see the change in energy

will be zero if $V(t+1) = -jV(t-1)$, which happens when a **cycle of length 8** is initiated.

B. Case 2 : M is Braided Skew Hermitian matrix

So, let $M = A - jA^T$ where A is a real matrix. Now it leads to $M^T = -j\bar{M}$ where \bar{M} is conjugate of M .

In this analysis, we will make use of the energy function defined for the fully parallel mode.

$$E_2(t) = \text{realpart}\{V(t-1)^+ * M * V(t)\}$$

Consider the change in energy,

$$E_2(t+1) - E_1(t) = \text{realpart}\{V(t)^+ * M * V(t+1) - V(t-1)^+ * M * V(t)\}$$

Let $V(t)^+ * M = C + jD$ where C and D are real matrices.

$$\begin{aligned} \Rightarrow V(t)^T * \bar{M} &= C - jD \\ \Rightarrow V(t-1)^+ * M * V(t) &= V(t)^T * M^T * \overline{V(t-1)} \\ \Rightarrow V(t-1)^+ * M * V(t) &= (C - jD) * (-j) * \overline{V(t-1)} \end{aligned}$$

By substituting $V(t+1) = V_R(t+1) + jV_I(t+1)$ and $V(t-1) = V_R(t-1) + jV_I(t-1)$, where V_R and V_I denote real and imaginary parts of V , we get

$$E_2(t+1) - E_2(t) = C * (V_R(t+1) + V_I(t-1)) - D * (V_I(t+1) - V_R(t-1))$$

The change in energy, $E_2(t+1) - E_2(t)$ will be zero when a cycle is initiated. As we can see the change in energy will be zero if $V(t+1) = jV(t-1)$, which happens when a **cycle of length 8** is initiated.

V. ARBITRARY COMPLEX RECURRENT HOPFIELD NETWORK: EXPERIMENTAL RESULTS

In this section, we will empirically study the dynamics of some Complex Recurrent Hopfield networks.

From now, the complex weight matrix will be decomposed as $M = A + jB$ where A and B are real matrices. Now, we will analyse the dynamics of the complex hopfield network with different possibilities of M .

Note that we already analysed 2 possibilities i.e. when A is symmetric and B is antisymmetric which leads to M being a Hermitian matrix in [5] and A is antisymmetric and B is symmetric which leads to M being a skewhermitian matrix in this paper. So, let us analyse the remaining possibilities. There are also many sub-cases in each possibility.

Now, the simulation procedure is as follows:

Here, we considered 20,000 instances where, in each instance, the number of neurons (in the range 4 to 65) and initial state and weight matrix are randomly chosen according to the case on which it is going to be tested. Now, for each of the below discussed cases and their sub-cases, we evaluated 20,000 instances and results are tabulated except for the sub-cases where cycle length is not bounded (very large numbers). Length of the cycle is represented as "L" in the tables.

A. A is symmetric and B is symmetric

Then this leads to M being an complex symmetric matrix i.e. $M^T = M$. Based on signs of elements of A and B , there are subcases in it.

(i) Now, If all entries of A and B are positive then in parallel mode, the cycle length of the network is bounded which means that the cycle length is a small number.

Number of instances	$L \leq 3$	$4 \leq L \leq 8$	$9 \leq L \leq 50$
20000	47	16827	3666

(ii) If entries of A and B are all negative then also the cycle length is bounded.

Number of instances	$L \leq 3$	$4 \leq L \leq 8$	$9 \leq L \leq 50$
20000	7373	9334	3293

(iii) Now, if signs of entries of A and B are arbitrary (either positive or negative), then in parallel mode, the cycle length of the network is not bounded by which it means that the cycle length is very large in many cases. Cycle lengths of about 10000 are also observed.

B. A is antisymmetric and B is antisymmetric

Then this leads to M being an complex antisymmetric matrix i.e. $M^T = -M$

(i) If all entries of M have both their real part and imaginary part as positive(or negative) then in parallel mode, the cycle length of the network is bounded.

No.of instances	$L \leq 3$	$4 \leq L \leq 8$	$9 \leq L \leq 50$	$51 \leq L \leq 100$
20000	499	3940	15411	150

(ii) If the entries of M are arbitrarily taken then the cycle length of the network is not bounded. Cycle lengths of about 10000 are also observed.

C. A is symmetric and B is asymmetric

(i) Now, if entries of A and B are all positive, then in parallel mode, the cycle length of the network is bounded. Majority

of the cycles are of length 1 which implies network reached to a stable state.

No. of instances	$L = 1$	$2 \leq L \leq 8$	$9 \leq L \leq 30$
20000	18981	133	886

(ii) If entries of A and B are all negative, then in parallel mode, the cycle length of the network is bounded and majority of the cycles are of length 2.

No. of instances	$L = 2$	$3 \leq L \leq 8$	$9 \leq L \leq 30$
20000	16080	885	3035

(iii) If entries of A and B are arbitrarily taken, then in parallel mode, the cycle length of the network is not bounded. Huge cycles are observed.

D. A is asymmetric and B is symmetric

(i) Now, if entries of A and B are all positive, then in parallel mode, the cycle length of the network is bounded. Majority of the cycles are of length 1 i.e. stable state is reached.

No. of instances	$L = 1$	$3 \leq L \leq 8$	$9 \leq L \leq 35$
20000	16334	490	3176

(ii) If entries of A and B are all negative, then in parallel mode, the cycle length of the network is bounded. Majority of the cycles are of length 2.

No. of instances	$L = 2$	$3 \leq L \leq 8$	$9 \leq L \leq 30$
20000	15773	1050	3177

(iii) If entries of A and B are arbitrarily taken, then in parallel mode, the cycle length of the network is not bounded. Huge cycles are observed.

E. A is antisymmetric and B is asymmetric

(i) If entries of B are arbitrarily taken and A is such that elements above the diagonal in A are all positive (or negative), then in parallel mode, the cycle length of the network is bounded. In 99% of the cases, cycles of length 4 are observed.

No. of instances	$1 \leq L \leq 3$	$L = 4$	$5 \leq L \leq 50$
20000	6	19484	510

(ii) If entries of A are arbitrarily taken and all entries in B are positive(or negative) then in parallel mode, the cycle length of the network is bounded. In 99% of the cases, cycles of length 4 are observed.

(iii) If entries of A and B are arbitrarily taken, then in parallel mode, the cycle length of the network is not bounded. Huge cycles are observed.

No. of instances	$1 \leq L \leq 3$	$L = 4$	$5 \leq L \leq 20$
20000	0	19997	3

F. A is asymmetric and B is antisymmetric

(i) If entries of A are arbitrarily taken and B is such that elements above the diagonal in B are all positive (or negative), then in parallel mode the cycle length of the network is bounded. Majority of the cycles are of length 2.

No. of instances	$L = 1$	$L = 2$	$3 \leq L \leq 50$	$51 \leq L \leq 150$
20000	683	16389	2871	57

(ii) If entries of B are arbitrarily taken and all entries in A are positive then in parallel mode, the cycle length of the network is bounded. Majority of the cycles are of length 1 which implies that a stable state is reached.

No. of instances	$L = 1$	$L = 2$	$3 \leq L \leq 5$
20000	19941	55	4

(iii) If entries of B are arbitrarily taken and all entries in A are negative then in parallel mode, the cycle length of the network is bounded. Majority of the cycles are of length 2.

No. of instances	$L = 1$	$L = 2$	$3 \leq L \leq 12$
20000	8	19990	2

(iv) If entries of A and B are arbitrarily taken, then in parallel mode, the cycle length of the network is not bounded. Huge cycles are observed.

VI. COMPLEX RECURRENT HOPFIELD NETWORK BASED ON STRUCTURED MATRICES: EXPERIMENTAL RESULTS

In this section, we will empirically study the dynamics of some Complex Recurrent Hopfield networks based on structured matrices like toeplitz and circulant matrices.

Here also, the complex weight matrix will be decomposed as $M=A+jB$ where A and B are real matrices. Now, will analyse the dynamics of the network through the following cases.

A. A is toeplitz matrix and B is toeplitz

This leads to M being a complex toeplitz matrix. An example matrix for $n=5$ is

$$\begin{bmatrix} 10 + j & 1 - 1j & -8 + 9j & -3 + 6j & 7 + 7j \\ 5 - 2j & 10 + j & 1 - 1j & -8 + 9j & -3 + 6j \\ -6 + 3j & 5 - 2j & 10 + j & 1 - 1j & -8 + 9j \\ -1 + 9j & -6 + 3j & 5 - 2j & 10 + j & 1 - 1j \\ -7 + j & -1 + 9j & -6 + 3j & 5 - 2j & 10 + j \end{bmatrix}$$

(i) If all entries of A and B are positive, then cycle length is bounded.

No. of instances	$1 \leq L \leq 4$	$5 \leq L \leq 8$	$9 \leq L \leq 25$
20000	5338	12688	1974

(ii) If all entries of A and B are negative, then cycle length is bounded.

No. of instances	$1 \leq L \leq 4$	$5 \leq L \leq 8$	$9 \leq L \leq 25$
20000	6787	11958	1255

(iii) If signs of entries of A and B are taken arbitrarily, then cycle length is not bounded. Huge cycle lengths are observed.

B. A is right circulant matrix and B is right circulant matrix

This leads to M being a complex right circulant matrix. An example matrix for n=5 is

$$\begin{bmatrix} 6 + 7j & -6 - 6j & -2 + 6j & 5 - 7j & -7 + 4j \\ -7 + 4j & 6 + 7j & -6 - 6j & -2 + 6j & 5 - 7j \\ 5 - 7j & -7 + 4j & 6 + 7j & -6 - 6j & -2 + 6j \\ -2 + 6j & 5 - 7j & -7 + 4j & 6 + 7j & -6 - 6j \\ -6 - 6j & -2 + 6j & 5 - 7j & -7 + 4j & 6 + 7j \end{bmatrix}$$

(i) If all entries of A and B are positive, then cycle length is bounded. Majority of the cycles are of length 1 and length 4.

No. of instances	$L = 1$	$2 \leq L \leq 3$	$L = 4$	$5 \leq L \leq 100$
20000	10190	12	9783	15

(i) If all entries of A and B are negative, then cycle length is bounded. Majority of the cycles are of length 2 and length 4.

No. of instances	$L = 1$	$L = 2$	$L = 4$	$5 \leq L \leq 25$
20000	6	9712	9794	30

(iii) If signs of entries of A and B are taken arbitrarily, then cycle length is not bounded. Huge cycle lengths are observed.

VII. APPLICATIONS

Traditionally, associative memories are defined by associating a single state (i.e. stable state) with the noise corrupted versions of it. In [4], the authors proposed the concept of "multi-state associative memories" by associating a initial condition with a cycle of states. Thus, the structured complex recurrent Hopfield network proposed in this research paper could be utilized as an interesting multi-state associative memory.

VIII. FUTURE WORK

- It is well known that complex numbers have polar representation as well as rectangular representation. In the above discussion, we have utilized rectangular representation of complex numbers and defined the Csignum function. We now propose a novel activation function denoted as NCsignum() and the associated complex valued neural networks. This function involves quantization of "magnitude" as well as phase of the net contribution at a neuron i.e. phase and magnitude are quantized to some "K" values. Phase quantization is already proposed in [10].

$$NCsignum_{\mathbb{K}}(a + ib) = re^{j\theta} \quad \text{where}$$

$$r = \begin{cases} 1 & \text{if } 0 \leq \text{Magnitude}(a + ib) < 1 \\ 2 & \text{if } 1 \leq \text{Magnitude}(a + ib) < 2 \\ \vdots & \\ K & \text{if } (K - 1) \leq \text{Magnitude}(a + ib) < K \end{cases}$$

Similarly, phase(θ) is also quantized.

- It is well known that "Quaternions" generalize the concept of "complex numbers". Thus, the state of a neuron in a novel complex, recurrent Hopfield network can be chosen to be a quaternion. The dynamics of such complex recurrent Hopfield networks can be studied as in the research paper. Also, Clifford algebra is a generalization of Quaternions. Thus, Clifford algebra based recurrent networks can also be investigated for their dynamics.

IX. CONCLUSION

In this research paper, an interesting structured complex recurrent Hopfield network (with skew Hermitian synaptic weight matrix) is proposed. In the parallel mode of operation of such a network, the dynamics is exactly predicted. Interesting application of such a network is discussed. Dynamics of a complex recurrent Hopfield based on Braided Hermitian and Braided Skew Hermitian is also predicted in parallel mode of operation. Experimental results for different types of Complex Recurrent networks are shown. Some applications are proposed.

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