Distributed Signal Processing : Neural Networks

by

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ABSTRACT

In this research paper, a novel non-linear distributed signal processing algorithm is proposed. By relating it to the Hopfield neural network, convergence of the algorithm is established. Also, linear consensus algorithm based on “hybrid topology” (locally centralized and globally distributed) is discussed.

1. Introduction:

Conventional signal processing deals with processing a signal at a single node/site in a centralized manner. In recent years, there has been increasing interest in processing signals in a distributed manner leading to the research area of “Distributed Signal Processing”. In such an effort, based on the initial condition, signals are processed at multiple sites in a distributed manner over the temporal dimension (time) and it is hoped that the signal processing algorithm converges.

One of the interesting distributed signal processing problems deals with the computation of average/mean in a distributed manner. Let there be \( N \) sensors measuring the temperature. The goal is to compute the average temperature over the sensor field i.e.

\[
\bar{y} = \frac{1}{N} \sum_{i=1}^{N} y_i.
\]

As shown in the following figures (Figure 1, Figure 2), there are two alternative approaches:

- Parallel (Centralized) computation of mean...Figure 1
- Distributed topology for computation of mean...Figure 2

Figure 1                                   Figure 2
Note: Throughout this research paper, we consider the application of Wireless Sensor Networks (WSNs). It is an example of Cyber Physical Systems [Lee]. It should be kept in mind that the distributed signal processing algorithms apply to other systems.

The parallel (centralized topology based) computation of mean is simple and straightforward. But in many applications such a topology is unrealistic. Thus, let us consider the distributed topology based computation of mean.

- **Distributed Topology: Consensus Algorithm:**

  The computation of mean is distributed over a collection of “N” nodes. The nodes are connected together with symmetric weights. The operation of iterative consensus algorithm is now discussed. For concreteness, we consider the case where there are three (3) nodes [KJM].

  (i) Let the distributed signal processing start at time ‘zero’ with all the nodes initialized with beginning values.

  (ii) For concreteness consider the network topology shown in Figure 3.

  ![Figure 3](image)

  Based on the estimates at time ‘n’ i.e. \( \{ x_j(n) \}_{j=1}^{3} \), we compute the estimates at time ‘n+1’ iteratively in the following manner:

  \[
  x_1(n+1) = W_{11}x_1(n) + W_{13}x_3(n) \\
  x_2(n+1) = W_{22}x_2(n) + W_{23}x_3(n) \\
  x_3(n+1) = W_{33}x_3(n) + W_{31}x_1(n) + W_{32}x_2(n)
  \]

  Incorporating the above computations into a vector-matrix form, we arrive at the following equation:
Thus, we arrive at the following equation:
\[
\begin{bmatrix}
x_1(n+1) \\
x_2(n+1) \\
x_3(n+1)
\end{bmatrix} =
\begin{bmatrix}
W_{11} & 0 & W_{13} \\
0 & W_{22} & W_{23} \\
W_{31} & W_{32} & W_{33}
\end{bmatrix}
\begin{bmatrix}
x_1(n) \\
x_2(n) \\
x_3(n)
\end{bmatrix}.
\]

From the above, starting at time zero, the vector at time ‘n+1’ can be expressed in the following form:
\[
\mathbf{x}(n+1) = W^{n+1} \mathbf{x}(0).
\]

To discuss the equilibrium / limiting behavior, we need the following notation
\[
\bar{x} = [1 \ 1 \ \ldots \ 1]^T, \text{ the all ones column vector.}
\]
\[
\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i(0).
\]

The following Theorem summarizes the limiting behavior under some sufficient conditions:

**Theorem 1 [ Xiao & Boyd, 04 ]:** If \( \bar{e}^T W = \bar{e}^T \), \( W\bar{e} = \bar{e} \) and spectral radius \( \rho \) i.e. \( \rho \left( W - \frac{1}{N} \bar{e} \bar{e}^T \right) < 1 \),

Then, consensus is achieved i.e.:
\[
\lim_{i \to \infty} \mathbf{x}(i) = \bar{x} \bar{e}
\]
and
\[
\lim_{i \to \infty} W^i = \frac{1}{N} \bar{e} \bar{e}^T.
\]

It is thus clear that for consensus to be achieved, it is sufficient that the symmetric weight matrix satisfies the Xiao & Boyd conditions.

Motivated by the research in Wireless Sensor Networks (and more generally Cyber Physical Systems [Lee]), the author formulated a distributed signal processing problem based on the occurrence / non-occurrence of event at the nodes. The details of such a problem are considered in Section 2. Also, motivated by the LEACH algorithm (and its variants in several applications), the author considered linear consensus algorithm implemented on “hybrid topology”. Details of the algorithm are discussed in Section 3. The research paper concludes in Section 4.

2. **Binary Sensed Measurements: Non-Linear Consensus Algorithm:**

**Hopfield Neural Network:**

In many real world applications (such as Wireless Sensor Networks / WSNs), the measurements at various nodes are processed using distributed signal processing algorithms. In traditional treatments (such
as [KJM]), it is assumed that the measurements i.e. \( \{x_i\}_{i=1}^N \) can assume any real values i.e. \( x_i \in \mathbb{R} \), the real number line.

- But in dealing with various interesting phenomena, the sensors (e.g. temperature) indicate the occurrence of an event or not (the central controlling base station is only interested whether an event occurred or not) i.e. \( x_i \in \{ \mp 1 \} \) (where \( x_i = +1 \) indicates that an event such as fire occurred and \( x_i = -1 \) indicates that no event has occurred).

- Also, the connections between nodes have the associated weights which are symmetric with respect to the nodes. Thus a symmetric weight matrix, \( \mathbf{W} \) specifies the connection structure of the collection of nodes that are monitoring the phenomenon in a cooperative manner.

- Also, each node has the associated threshold value. Thus, a threshold vector, \( \mathbf{T} \) summarizes the set of threshold values at all the nodes.

- The consensus algorithm that is implemented is non-linear. The state at each node is updated in the following manner:
  \[
  x_i(n + 1) = \text{Sign} \left\{ \sum_{j=1}^N W_{ij} x_j(n) - T(i) \right\}, \quad ...(2.1)
  \]
  where \( \text{Sign}(.) \) denotes the Signum function. This state updating scheme has natural interpretation in terms of determining the evolution of occurrence of events at various nodes of the network.

- From the above state updating scheme (of the non-linear consensus algorithm), it is clear that the state of the network of ‘\( N \)’ signal processing nodes is the \( N \)-dimensional unit hypercube.

  Now we summarize the operation of Hopfield Neural Network (Associative Memory) to understand the relationship to the above distributed signal processing problem [BrG].

- **Hopfield Neural Network:**
  Hopfield neural network is a non-linear discrete time system that can be represented by a weighted and undirected graph. Each edge of the graph has a weight attached to it and a threshold value is attached to each node (artificial neuron) of the graph. The number of nodes of the graph is denoted as the “order” of the network. Let ‘\( L \)’ be a neural network of order ‘\( N \)’. Then L is uniquely specified by the tuple (\( \mathbf{W}, \mathbf{T} \)) where
  - \( \mathbf{W} \) is an \( N \times N \) symmetric matrix, where \( W_{ij} \) is equal to the weight attached to edge \((i,j)\);
• T is a vector of dimension N, where \( T_i \) denotes the threshold attached to node ‘i’.

The state of each possible artificial neuron can be +1 or -1. The state of a node at time ‘n’ is denoted by \( X_i(n) \). Thus the state vector of the Hopfield neural network at time ‘n’ is denoted by \( \bar{X}(n) \).

The state at node ‘i’, at the time instant ‘n+1’ is computed in the following manner:

\[
x_i(n+1) = \text{Sign} \{ \sum_{j=1}^{N} W_{ij} x_j(n) - T(i) \} \quad \ldots(2.2)
\]

The state vector at time ‘n+1’ i.e. \( \bar{X}(n+1) \) is computed from the current state vector at time ‘n’ by performing the computation in (2.2) at a set S of the nodes of the network. The modes of operation of Hopfield network is determined by the method by which the set S is selected at each time interval.

• If the computation in (2.2) is performed at a single node at any time interval i.e. \(|S|=1\), then we say that the network is operating in serial mode.

• On the other hand, if \(|S|=N\), then we say that the network is operating in a fully parallel mode.

• All other cases i.e. \(1<|S|<N\), will be called parallel modes of operation.

• Definition: A state \( \bar{X}(n) \) is called STABLE if and only if

\[
\bar{X}(n) = \text{Sign} \{ W \bar{X}(n) - T \} \quad \ldots(2.3)
\]

i.e. no change occurs in the state of the network regardless of the mode of operation of the artificial neural network.

• An important property of the Hopfield neural network is that it will always converge to a stable state when operating in the serial mode and to a cycle of length atmost two when operating in a fully parallel mode. Thus, we have the formal convergence Theorem.

• Theorem 2: Let \( L = (W,T) \) be a Hopfield neural network, with \( W \) being a symmetric matrix; then the following hold true.

(A) Hopfield: If \( L \) is operating in a serial mode and the elements of the diagonal of \( W \) are non-negative, the network will always converge to a stable state (i.e. there are no cycles in the state space).

(B) Goles: If \( L \) is operating in a fully parallel mode, the network will always converge to a stable state or to a cycle of length 2 (i.e. the cycles in the state space are of length \( \leq 2 \)).
On comparing the above non-linear consensus algorithm with the dynamics of Hopfield neural network, we realize that the state updation scheme in (2.2) corresponds to the serial mode of operation of associated Hopfield neural network. Thus, the non-linear consensus algorithm discussed above always converges (serial mode of operation).

Once the state vector (corresponding to all the nodes) converges to a stable state, a majority vote is taken (over the occurrence of an event or not) to determine if the event is declared over the sensor field. Thus, a deterministic decision (with respect to the occurrence of an event over the sensor field) can always be attained when the number of processing sensor nodes is “odd”. Hence the goal of non-linear consensus algorithm is achieved.

Full implications of the convergence Theorem (Theorem 2) to consensus algorithms need to be explored [Rama1].

3. Hybrid Topology: Cyber Physical Systems: Linear Consensus Algorithm:

In the recent years, many applications motivated the design of distributed signal processing algorithms (for instance smart grid design). There were two main efforts:

- Centralized computation of say a quantity like “mean”.
- Decentralized computation of say “mean” (using the “consensus” algorithm).

To the best of our knowledge, researchers in distributed signal processing have only considered the above two possible cases. The author realized (during the invited talk of Jose Moura [Mou]) that in many cyber physical systems (such as wireless sensor networks) the computation of interesting quantity (say “mean” values of the associated natural/artificial phenomenon) is “partly distributed” and “partly centralized”.

- In more clear terms, researchers have not taken into account “hybrid topology” (for the processing nodes), in which certain nodes are connected together in a centralized fashion and the associated “processing centers” are connected in a distributed topology. Such a topology naturally arises in many applications of distributed signal processing. For instance, LEACH (Low Energy Adaptive Clustering Hierarchy) algorithm in Wireless Sensor Networks (WSNs) falls in this category. It is briefly discussed below to illustrate interesting features of hybrid topology.

- LEACH algorithm:

  In the case of Wireless Sensor Networks (WSNs), wireless sensor nodes are deployed over the sensor field. They collectively monitor the phenomenon and send the sensed measurements to the Base Station (BS). LEACH algorithm was designed for WSNs where sensors and the Base Station are stationary.
In this algorithm, the wireless sensors locally communicate with each other and form clusters with the associated “cluster heads” (group leaders). Typically, cluster head is the node with the highest residual energy. It locally collects the sensed readings (from the members of the cluster) and sends the readings to the Base station with the help of other cluster heads.

Thus, from the distributed signal processing viewpoint, within a cluster “centralized” topology based local computation takes place. Also, across the cluster heads, “distributed” topology based global computation takes place.

- It is clear that the topology in which processing nodes are connected determines the connection matrix $W$.

- When distributed topology is considered and linear consensus algorithm is utilized, the sufficient conditions specified by Xiao & Boyd ensure that consensus is achieved (i.e., convergence of the algorithm to the estimate provided by the centralized topology).

**Constraints on Hybrid Topology:**

- In view of Xiao & Boyd sufficient conditions, we consider the following hybrid topology which ensures that “consensus” (desired convergence) is achieved.

- Each node in a cluster of size $M$ (i.e., there are $M$ nodes in the group) is connected to the “cluster head” with weight $\frac{1}{M}$. They are not directly connected to one another. Also, each node is connected to the cluster heads of other clusters (M-1 of them i.e., We assume that the total number of nodes $N = M^2$) with weight $\frac{1}{M}$.

- Each cluster head is connected to itself with weight $\frac{1}{M}$ and it is also connected to (M-1) other cluster heads with weight $\frac{1}{M}$.

- The hybrid topology with three (3) clusters is illustrated in Figure 4.

- **Lemma 1:** With the above hybrid topology, the $W$ matrix satisfies the Xiao & Boyd conditions. Hence the associated linear consensus algorithm converges to the desired solution.

**Proof:** Follows from routine verification of Xiao & Boyd conditions Q.E.D.

**Remark:**

- Thus, the algorithms (such as statistical estimation techniques using the concepts such as “innovations” process) developed for purely centralized/purely decentralized case should be modified for the case that naturally arises in the design of many cyber physical systems [Rama2].
4. Conclusions:

In this research paper, a non-linear consensus based distributed signal processing algorithm is proposed. By relating it to Hopfield neural network, convergence of the algorithm is discussed. Also, a linear consensus algorithm on “hybrid network topology” is discussed.

References:


