A few limits to Quantum Information Theory

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To my eccentric family,
whose genes i share

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CERTIFICATE OF AUTHORSHIP

I, Subhasree Patro, declare that the thesis, titled "A few limits to Quantum Information Theory", and the work presented herein are my own. I confirm that this work was done wholly or mainly while in candidature for a research degree at IIIT-Hyderabad.

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It is certified that the work contained in this thesis, titled "A few limits to Quantum Information Theory" by Subhasree Patro, has been carried out under my supervision and is not submitted elsewhere for a degree.

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Abstract

Quantum superposition and entanglement (a special case of superposition) play a pivotal role in achieving information processing tasks which could otherwise be not possible through any classical resource. It is further interesting to note that these very properties constrict us from performing some tasks which are achievable classically.

As quantum entanglement has proven to be a key resource for various information theoretic tasks, its numerous applications makes it imperative to study its behaviour. Considerable studies have been done on how to generate entanglement, the chief of which is by using global unitary operations. The thesis further examines the limits imposed on quantum information theory due to the probability structure of mixed states. In mixed states the probability occurs at two levels, first it starts as a description of the system as a statistical mixture, second as the intrinsic superposition that is contained by each entity of the ensemble. We witness how the global unitaries, that can make any pure separable state entangled, fail to generate entanglement for a class of mixed states. The discussion begins with the definition of separable states and non-local states. It is then followed by a concise study on the ***absoluteness*** of the above mentioned class of states. Following which is an extensive study on ***absoluteness*** of non-negative conditional von Neumann entropy, which happens to be the decisive part of the second chapter.

The thesis concludes by discussing another limit that superposition imposes on quantum theory. The impossibility to clone coherence of an arbitrary state. It also addresses the logical implication of no-cloning of coherence which is the famous no-cloning theorem.

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Chapter 1

Foundations of Quantum Theory

> "To reject one paradigm without simultaneously substituting another is to reject science itself."
>
> – *Thomas S. Kuhn*

The quantum mechanical formalism that we currently hold is a culmination of many scientific developments that took place in the early 20th century. The theory was not developed in a strictly logical way, instead, it was a series of guesses inspired by profound physical insights of scientists like Max Planck, Louis de Broglie, Neils Bohr, Werner Heisenberg, Erwin Schrodinger, Albert Einstein, Max Born, Paul Dirac and others. Therefore, to be able to appreciate the current mathematical formulations of quantum mechanics better, it becomes important that we briefly recount the historical developments that lead to it.

Hence, the chapter starts with the historical overview of quantum mechanical formalism in section 1.1. After which the quantum mechanical formalism along with the postulates of quantum theory are discussed concisely in section 1.2. This text assumes that the reader is familiar with the basics of linear algebra. Following that, in section 1.3 and 1.4 the two very important milestones in the history of quantum theory, namely the EPR paradox and Bell's Inequality are discussed so as to provide an understanding of the significance of Bell's theorem and thus the Bell's inequalities which are relevant for a fuller appreciation of the second chapter. After that a short review on Shannon and von Neumann entropies, along with their properties are discussed in section 1.5 because the results presented in the second chapter are built upon these fundamental theories. Finally, in section 1.6, a concise introduction to entanglement and its usage as a resource in several information theoretic tasks are discussed.

1.1 Historical overview

At the end of nineteenth century, matter was understood to come in two forms, particles or waves. The departure from classifying matter into this neat division of wave-like and particle-like forms, is the story of quantum theory.

The journey of quantum mechanics is closely associated with the quest for uncovering the true nature of light. In the late seventeenth century, Isaac Newton formulated the corpuscle theory of light. Later, Christian Huygens in 1690 proposed and Thomas Young in 1801 demonstrated that light behaved like a wave. Another puzzle that bothered scientists in the late 19th century was the black body radiation. One of the major failings of classical theory of light was that it predicted that certain objects, namely black bodies, would radiate an infinite amount of energy. But the experiments suggested otherwise. Later in the year 1900, Max Planck proposed that light was emitted in quantized lumps of energy that come as integral multiples of $E = h\nu$, where h is the Planck's constant. Planck believed that the rule for the absorption and emission of light in quantised lumps of energy applied only to black body radiation was a property of atoms, but, Einstein saw that it was a property pertaining to electromagnetic radiation. Whether it was a radiation from the black body or any other origin, it did not matter what the source was. This proposal was a result of his work on the photoelectric effect which deals with the absorption of light and the emission of elections from a material. In 1913, Niels Bohr stated that electrons in hydrogen atoms have wavelike properties. About a decade later in 1924, Louis de Broglie proposed that all particles are associated with waves, which in 1927 was experimentally verified by Clinton Davisson and Lester Germer. This gave rise to the theory of wave-particle duality. Meanwhile, in 1925, Werner Heisenberg formulated a version of quantum mechanics using two equivalent mathematical formalisms, matrix mechanics and wave mechanics. Subsequently, in the year 1926, Erwin Schrodinger formulated a version of quantum mechanics that was based on waves. In 1926, Max Born correctly interpreted Schrodinger's wave as a probability amplitude. Other than developing language to go along with the theory, there were some hurdles, like measurement and wave function collapse that were addressed in different interpretations in quantum theory. One of the most widely accepted interpretation was the Copenhagen interpretation which was developed and proposed by Bohr and Heisenberg.

The mathematical formalism and results in this thesis align with the Copenhagen interpretation of quantum mechanics, and no attempt to challenge that is made here.

1.2 Mathematical Formulations of Quantum Mechanics

Mathematical formulation of quantum mechanics was majorly developed by Paul Dirac, David Hilbert, John von Neumann and Hermann Weyl. It is based on the framework of linear algebra and functional analysis. Any state of the quantum mechanical system is called a state vector, they are unit vectors and they reside in complex Hilbert space. In this section we will walk through the state space of a quantum system, the postulates of quantum and the matrix formalism of the state of a system.

1.2.1 Hilbert spaces

A Hilbert space $\mathcal{H}$, is a vector space Φ defined over a field of complex numbers $\mathbb{C}$, with a sesquilinear inner product defined, that obeys the following rules,

1. **Vector Addition**: For vector elements $\varphi, \psi, \chi \in \Phi$, it follows

$$\begin{aligned} \varphi + \psi \in \Phi & \qquad \text{(Closure)} \\ \varphi + \psi = \psi + \varphi & \qquad \text{(Commutative)} \\ (\varphi + \psi) + \chi = \varphi + (\psi + \chi) & \qquad \text{(Associative)} \end{aligned} \tag{1.1}$$

   Also define zero vector, $\mathbf{0}$, such that $\varphi + \mathbf{0} = \mathbf{0} + \varphi = \varphi$.

2. **Scalar Multiplication**: For scalars $a, b \in \mathbb{C}$ and vectors $\varphi, \psi \in \Phi$,

$$\begin{aligned} a.\varphi \in \Phi & \qquad \text{(Closure)} \\ a(b.\varphi) = (ab).\varphi\,, \quad 0.\varphi = \mathbf{0}\,, \quad 1.\varphi = \varphi & \qquad \text{(Identity)} \\ (a+b)\varphi = a\varphi + b\varphi\,, \quad a(\varphi + \psi) = a\varphi + a\psi & \qquad \text{(Distributive)} \end{aligned} \tag{1.2}$$

3. **Inner Product**: Hilbert space is equipped with *inner product*, which is a function defined as $(*, *) : \Phi \times \Phi \to \mathbb{C}$, mapping two vectors $\varphi, \psi \in \Phi$ to a complex number with the

following properties,

$$
\begin{aligned}
(\varphi, \varphi) \geq 0 \text{ and } (\varphi, \varphi) = 0 \iff \varphi = 0 & \qquad \text{(Positive-definite)} \\
(\varphi, \psi) = (\psi, \varphi)^* & \qquad \text{(Conjugate Symmetry)} \\
(\varphi, a\psi) = a(\varphi, \psi) \text{ and,} & \\
(\varphi, \psi_1 + \psi_2) = (\varphi, \psi_1) + (\varphi_2, \psi_1) & \qquad \text{(Linear in second arguement)}
\end{aligned} \tag{1.3}
$$

Dirac or *bra* − *ket* notation is the standard notation used to describe quantum states. We denote the vectors by $|\psi\rangle$, this is called a *ket* and $\langle\psi|$, called a *bra* is the complex conjugate of $|\psi\rangle$. Two vectors $|\psi\rangle$ and $|\varphi\rangle$ are said to be orthogonal iff their inner product, i.e. $\langle\psi|\varphi\rangle = 0$.

1.2.2 Postulates of Quantum Mechanics

Quantum Mechanics is an axiomatic theory. In this section, we will discuss the postulates that establish the mathematical description of the system and its time evolution. In addition to that we will also address the measurement process and it's effect on the system.

Postulate 1 Any isolated physical system is described by a complex Hilbert space, known as the state space of the system. The system is completely described by its state vector, denoted by $|\psi\rangle$ which is a unit vector in the system's state space [2].

The simplest quantum system is called as a *qubit*. It has a two-dimensional state space. A qubit is described by $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $|0\rangle$ and $|1\rangle$ form the orthonormal basis for the state space, and, α and β are complex numbers with $|\alpha|^2 + |\beta|^2 = 1$ because $|\psi\rangle$ has to be a normalised vector.

A bit can either take 0 or 1, unlike a qubit which is generally a superposition of the two.

The general d-dimensional state space is described by a *qudit*, $|\phi\rangle = \sum_i^d \alpha_i|i\rangle$.

Postulate 2 The evolution of a closed quantum system is described by (time-dependent) unitary transformation. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi^{'}\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2, $|\psi^{'}\rangle = U|\psi\rangle$ [2].

The postulate 2 demands that the described system is closed and is not interacting with any other systems. Unitary evolutions ensure that magnitude of the state vector is preserved.

Postulate 3 Quantum measurements are described by a collection $\{M_m\}$ of measurement operators. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by [2],

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle, \tag{1.4}$$

and, the state after measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}}. \tag{1.5}$$

The measurement operators fulfill the completeness condition,

$$\sum_m M_m^\dagger M_m = I. \tag{1.6}$$

The completeness condition expresses the fact that

$$\sum_m p(m) = \sum_m \langle\psi|M_m^\dagger M_m|\psi\rangle = 1. \tag{1.7}$$

In quantum mechanics, every observable is a hermitian operator because we need the expectation value of the operator to be real. Also, after the measurement the state collapses to one of the eigen states of hermitian operator.

Postulate 4 The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes ... \otimes |\psi_n\rangle$ [2].

Tensor product is the mathematical structure that is used to describe composite physical systems. This is a canonical way of describing composite systems in quantum mechanics.

1.2.3 Density operator

The density operator or more commonly known as matrix formalism is equivalent to the state vector formalism but has its advantages when there is incomplete information about the state of the system that is being described.

Suppose we have an ensemble of states where we know that all the states of the ensemble are one of the same then we call this state as a ***pure state***. Otherwise, the state is a statistical mixture of different states from the ensemble, and is referred to as ***mixed state***.

1.2.3.1 Density operator for pure states

Consider a pure state $|\psi\rangle = \sum_i \alpha_i |i\rangle$, where $\{|i\rangle\}$ comprises an orthonormal basis. Here, $\sum_i |\alpha_i|^2 = 1$ as $|\psi\rangle$ is a normalized state. The density operator is then written as the projector $\rho = |\psi\rangle\langle\psi|$ on the vector $|\psi\rangle$. The matrix elements of

$$[\rho]_{ij} = \langle i|\rho|j\rangle = \langle i|\psi\rangle\langle\psi|j\rangle = \alpha_i \alpha_j^* \tag{1.8}$$

The normalization condition, $\sum_i |\alpha_i|^2 = 1$ ensures that,

$$\sum_i |\alpha_i|^2 = \sum_i [\rho]_{ii} = tr(\rho) \tag{1.9}$$

The expectation value of an observable $\hat{O}$, will be

$$\langle\hat{O}\rangle = \langle\psi|\hat{O}|\psi\rangle = \sum_{ij} \alpha_i \alpha_j^* [\hat{O}]_{ji}, \tag{1.10}$$

where $[\hat{O}]_{ji}$ are the matrix elements of the observable $\hat{O}$, then

$$\langle\hat{O}\rangle = \sum_{ij} \langle j|\rho|i\rangle\langle i|\hat{O}|j\rangle = \sum_j \langle j|\rho\hat{O}|j\rangle = tr(\rho\hat{O}) \tag{1.11}$$

Note that we have used the completeness condition $\sum_i |i\rangle\langle i| = 1$

We can derive some properties of pure states from the definition above,

1. Density matrix is ***hermitian***, i.e. $\rho = \rho^\dagger$. This property applies to density matrices of mixed states as well.
2. $\rho^2 = |\psi\rangle\langle\psi||\psi\rangle\langle\psi| = \langle\psi|\psi\rangle|\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = \rho$, as $|\psi\rangle$ is normalised.
3. $tr(\rho^2) = tr(\rho) = 1$.

1.2.3.2 Density operator for mixed states

Density matrix of statistical mixture of an ensemble of pure states $\{|\psi_k\rangle\}$ is given by,

$$\rho = \sum_k p_k |\psi_k\rangle\langle\psi_k|, \sum_k p_k = 1. \tag{1.12}$$

The idea behind this definition is that, ρ is prepared by picking the state $|\psi_k\rangle$ with probablity p_k. It is easy to verify that,

$$tr(\rho) = \sum_k p_k tr(|\psi_k\rangle\langle\psi_k|) = \sum_k p_k = 1. \tag{1.13}$$

The expectation value of any observable $\hat{O}$ acting on the state can easily be shown to be $\langle\hat{O}\rangle = tr(\rho\hat{O})$, the derivation is in the same lines to that of pure states.

An important thing to note here is that as for mixed states $\rho^2 \neq \rho$, $tr(\rho^2) < 1$. This condition can be used to distinguish between a pure and a mixed state.

1.2.3.3 Reduced density operator and partial trace

A description of a subsystem of a composite system is provided by the ***reduced density operator*** [2]. Suppose we have a composite system ρ_{AB} comprising of subsystems A and B. Then the reduced density operator is defined as,

$$\rho_A = tr_B(\rho_{AB}), \tag{1.14}$$

where tr_B is known as the map of operators known as the ***partial trace*** over subsystem B. The partial map is defined by

$$tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) = |a_1\rangle\langle a_2| tr(|b_1\rangle\langle b_2|) = \langle b_2|b_1\rangle|a_1\rangle\langle a_2|, \tag{1.15}$$

The physical justification of making this map is that the reduced density matrix provides the correct measurement statisctics of measurement made on system A [2]. When a composite system is expressed as tensor product, $\rho_{AB} = \rho \otimes \sigma$ of its subsystems, then

$$\rho_A = tr_B(\rho_{AB}) = tr_B(\rho \otimes \sigma) = \rho tr(\sigma) = \rho, \tag{1.16}$$

which is a result that we intuitively expect.

1.2.3.4 Bloch sphere, a geometric visualisation of all single qubit states

The Bloch sphere is three dimensional unit sphere which gives the geometric representation of states in $\mathbb{C}^2$ state space. All the states are rays originating from the center of the sphere, with the pure states lying on the surface while mixed states making up the interior of the sphere [2].

Every pure state on the Bloch sphere is described by two parameters, $\theta \in [0, \frac{\pi}{2}]$ and $\phi \in [0, 2\pi)$. Every mixed state on the Bloch sphere can be expressed as $\rho = \frac{I + \vec{m}.\vec{\sigma}}{2}$ with $\vec{m} = (m_x, m_y, m_z)$ as the Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

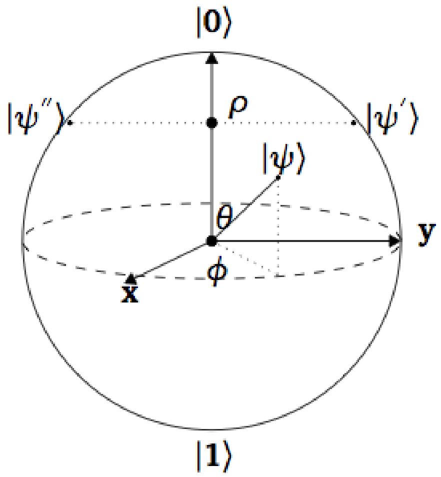

Figure 1.1: This figure represents the Bloch sphere representation of single qubit states. The states $|\psi\rangle$, $|\psi^{'}\rangle$ and $|\psi^{''}\rangle$ are pure states lying on the surface of the Bloch sphere. The state ρ represents a mixed state. This lies inside the Bloch sphere.

One interesting observation is that, in mixed states the probabilities occur at two levels, first as a description of the system as a statistical mixture and second as the intrinsic superposition that each entity of the ensemble contains. Given a density matrix, it is impossible to quantify how much interference is due to the statistical mixture and how much is due to the intrinsic quantum nature, unless one knows the elements of the ensemble. One can easily visualize it using an example. The state ρ in the Fig. 1.1 can be written in infinitely many ways. The state $\rho = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1| = \frac{1}{2}|\psi^{'}\rangle\langle\psi^{'}| + \frac{1}{2}|\psi^{''}\rangle\langle\psi^{''}|$. Clearly there is a mismatch in the statistical probability distribution for the two mentioned ensembles.

1.3 EPR paradox

In their critical work in 1935, Einstein, Podolsky, Rosen argued that the description of reality provided by the quantum theory was incomplete [3]. They defined an ***element of physical reality*** associated with a physical quantity as existing if its value can be predicted with certainty while not disturbing the system in any way. This was known as the ***Reality Criterion (R)***. They also defined ***Locality Criterion (L)***, which stated that elements of reality of one system are not affected by measurements performed on another space-like separated system.

They considered a two-qubit singlet state

$$|\psi_{singlet}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{1.17}$$

For the given state $|\psi_{singlet}\rangle$, we can easily compute the conditional probability of finding the measurement outcome of the subsystem B given the measurement of outcome of the subsystem A.

Let the measurement operator used here is σ_z, whose eigen states are $|0\rangle$ and $|1\rangle$. The conditional probability of finding $\sigma_{Bz} = +1$, given that $\sigma_{Az} = -1$ is

$$Prob(\sigma_{Bz} = +1|\sigma_{Az} = -1) = \frac{Prob(\sigma_{Bz} = +1, \sigma_{Az} = -1)}{Prob(\sigma_{Az} = -1)} = \frac{1/2}{1/2} = 1. \tag{1.18}$$

Similarly,

$$Prob(\sigma_{Bz} = -1|\sigma_{Az} = +1) = 1. \tag{1.19}$$

Therefore, we see that measuring σ_{Az} enables one to predict the outcome of σ_{Bz} deterministically. Lets say at time t_1 one measures σ_{Az}, then one can predict the result of σ_{Bz} at any time $t_2 > t_1$. Using the reality criterion R, we can say that $[\sigma_{Bz}]$, the measurement outcome, exists at any time $t_2 > t_1$. Suppose we measured the subsystem A and we got the outcome $+1$. Then according to the postulates of quantum mechanics the joint system collapses to $|\psi^{'}\rangle = |01\rangle$. And assuming that there is no interaction with any other system, the joint system remains as $|\psi^{'}\rangle = |01\rangle$ at any time $t_2 > t_1$. This corresponds to the fact that at any time $t_2 > t_1$ we can predict the measurement result of σ_{Bz} to be -1 and we can say the system B is in an eigen state of σ_{Bz} at any time $t_2 > t_1$. Hence, we infer by R the existence of $[\sigma_{Bz}] = -1$. Then one can argue that system B was in an eigen state of σ_{Bz} at a time $t_3 < t_1$ by invoking the locality criterion L. But $|\psi_{singlet}\rangle$ at a time t_3 is certainly not in an eigen state of σ_{Bz}. Hence a contradiction.

We can summarize the EPR paradox saying that the assumption of *Reality criterion*, R and the *Locality criterion*, L leads to the *incompleteness* of quantum theory. Therefore, the following logical statement summarizes the argument that using the quantum formalism,

$$R \wedge L \rightarrow Incompleteness. \tag{1.20}$$

This means that either the assumption R or L or both is wrong, or quantum theory itself is incomplete. Einstein, Podolsky, Rosen strongly believed it was the later.

1.4 Hidden variable theory and Bell's inequality

The conclusion that was reached by EPR was that quantum theory was incomplete. Therefore, using the quantum mechanical theory one cannot determine the results of measurements on the

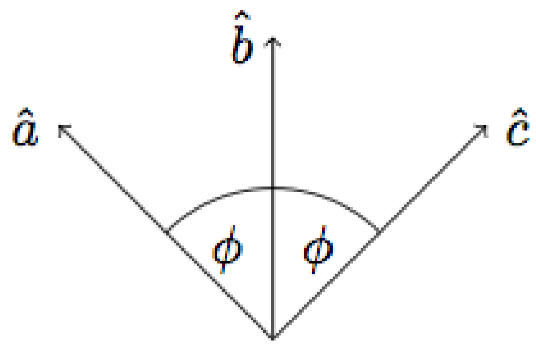

Figure 1.2: The figure depicts the directions of $\vec{a}$, $\vec{b}$ and $\vec{c}$ chosen

individual systems. Then it must be that there exists some deeper description that quantum mechanics was not able to capture. This gave rise to alternate theories, called as ***Hidden variable theories*** (HVTs). These theories attempted to complete the quantum mechanical description using additional parameters called the hidden variables, λ. In this section λ is taken to be as unit vector defined with uniform probability distribution. The assumption is that the hidden variables would contain the necessary information about the measurement and their outcomes.

In this section we will discuss an example of an HVT model given for a general single qubit state. Any general single qubit state ρ can be written as $\rho = \frac{\mathrm{I}+\vec{m}.\vec{\sigma}}{2}$. Given a measurement operator $\hat{A}(\vec{a}) = \vec{a}.\vec{\sigma}$, the expected value of measurement of ρ along the direction $\vec{a}$ according to quantum mechanics would be

$$\left\langle \hat{A}(\vec{a}) \right\rangle = \vec{m}.\vec{a}. \tag{1.21}$$

Here $\hat{A}(\vec{a}) \in \{+1, -1\}$. As according to HVT models the measurement outcome is determined by both $\vec{a}$ and by $\vec{\lambda}$, we will rewrite $\hat{A}(\vec{a})$ as $\hat{A}(\vec{a}, \vec{\lambda}) = \pm 1$. As the unit vector $\vec{\lambda}$ is uniformly distributed over a sphere, the $\rho(\vec{\lambda})d(\vec{\lambda}) = \frac{1}{4\pi} \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$. Using a measurement description of a choice, say [4] the outcome of the measurement on ρ can be deterministically computed as follows,

$$\hat{A}(\vec{a}, \vec{\lambda}) = \mathrm{sign}\left[\left(\vec{m} - \vec{\lambda}\right) \cdot \vec{a}\right]. \tag{1.22}$$

Averaging over $\vec{\lambda}$ gives the expectation value:

$$\left\langle \hat{A}(\vec{a}, \vec{\lambda}) \right\rangle = \int \mathrm{d}\vec{\lambda} \rho(\vec{\lambda}) \hat{A}(\vec{a}, \vec{\lambda}) = \vec{m} \cdot \vec{a}, \tag{1.23}$$

which reproduces the quantum mechanical statistics for a single qubit state exactly [4].

Following a similar line of thought, John Bell in 1964 derived an inequality based on a measurement in three arbitrary directions on a singlet state ρ (as mentioned in Eq. 1.17). Let us denote the three arbitrary directions by $\vec{a}$, $\vec{b}$ and $\vec{c}$. The expectation value of joint measurement

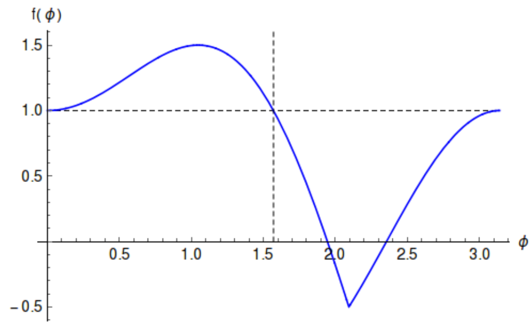

Figure 1.3: The figure depicts the ploting of $f(\phi)$ for $\phi \in [0, 180°]$

in direction $\vec{a}$ and $\vec{b}$ on a singlet state ρ is given as,

$$E_{HV}(\hat{a}, \hat{b}) = \int d\lambda \rho(\lambda) \hat{A}(\hat{a}, \lambda) \hat{B}(\hat{b}, \lambda), \tag{1.24}$$

where the individual measurements $\hat{A}(\vec{a}, \vec{\lambda})$, $\hat{B}(\vec{b}, \vec{\lambda})$ have only two possible values $\{+1, -1\}$.

For a successful HVT, the expectation value should match the expectations obtained from quantum mechanics, and, the expectation value from quantum mechanics is given by,

$$E_{QM}(\hat{a}, \hat{b}) = -\hat{a}.\hat{b}, \tag{1.25}$$

The hidden variable λ follows uniform probability distribution over the sphere, and is normalised so that

$$\int d\lambda \, \rho(\lambda) = 1. \tag{1.26}$$

From Eq.(1.24) and $\hat{A}(\vec{a}, \vec{\lambda}) = \pm 1$, $\hat{B}(\vec{b}, \vec{\lambda}) = \pm 1$, with some manipulation the Bell inequality, is obtained [5]:

$$1 + E_{HV}(\hat{b}, \hat{c}) \geq \left| E_{HV}(\hat{a}, \hat{b}) - E_{HV}(\hat{a}, \hat{c}) \right| . \tag{1.27}$$

For a choice of directions of $\vec{a}$, $\vec{b}$ and $\vec{c}$ as shown in Fig. 1.2, the inequality in Eq. 1.27 becomes

$$f(\phi) = \cos\phi + |\cos\phi - \cos 2\phi| \leq 1 \, . \tag{1.28}$$

By plotting these values of $f(\phi)$ for $\phi \in [0, 180°]$ one can see that Bell inequality is violated for a range of $\phi \in [0, 90°]$, with the maximum violation at $\phi = 60°$ as shown in the Fig. 1.3. Hence, proving that not all hidden variable models are compatible with the predictions of quantum mechanics. John Bell also showed that violation of Bell's inequality meant giving up factorizability in the joint probability distributions, which when given up means that measurement on one particle of the singlet state is statistically dependent on the probability distribution of measurements on the second particle.

1.5 Quantifying classical and quantum information

In 1948, Claude E. Shannon developed the first successful theory of Information. He modeled information as events that occurred with certain probabilities [6]. This measure information was later known as entropy. His subsequent papers on this subject laid the groundwork for everything that is being done in Information theory.

1.5.1 Classical Information Theory

Shannon postulated the following requirements that any measure of classical information should follow [6].

1. The amount of information I in an event x must only depend on its probability p. Therefore, information $I(p)$ should be a function on the probability p.

2. $I(p)$ should be a continuous function, i.e., a small change in probability should lead to a small change in the information contained.

3. $I(p_x, p_y) = I(p_x) + I(p_y)$, this assumption of additivity is required to understand the amount of information contained in a system with two independent events. The information gained in both these events together, $I(p_x, p_y)$, is the sum of the information gained in each of the two events, $I(p_x) + I(p_y)$.

Please note that there exists some measures of information that do not have follow all these requirements, specially the additivity condition.

1.5.1.1 Shannon Entropy

The *Shannon entropy* is the key concept of classical information theory. This entropy function on a random variable X aims to capture the amount of information that is gained in an average if we learn the value of X. It is defined as the negative logarithmic function of the probability distribution associated with the random variable X, [7, 8, 6]

$$H(X) = -\sum_x p(x) \log p(x), \tag{1.29}$$

where $p(x)$, is the probability distribution associated with the random variable X.[1] Also, joint entropy of two random variables X and Y are given by

$$H(X,Y) = -\sum_{x,y} p(x,y) \log p(x,y), \tag{1.30}$$

where $p(x,y)$, is the probability distribution associated with the random variable X and Y jointly. It is easy to notice that each term that contributes to the Shannon entropy is non-negative. Hence, Shannon entropy is always non-negative quantity and is zero when the occurrence of an event is definite.

Shannon entropy is very useful in communication theory but discussing that is not in the scope of this thesis.

1.5.1.2 Conditional Entropy

The *Conditional entropy* is the entropy of a random variable X given that we know the value of Y [2]. If for a given value of $Y = y$, the entropy of X is $H(X|Y = y)$, then the conditional entropy $H(X|Y)$ is the weighted average [8]:

$$\begin{aligned} H(X|Y) &= \sum_y p(y) H(X|Y=y) = -\sum_y p(y) \sum_x p(x|y) \log p(x|y) \\ &= -\sum_{x,y} p(x,y) \log p(x|y) = \sum_{x,y} p(x,y) \log \frac{p(y)}{p(x,y)} \\ \Rightarrow H(X|Y) &= H(X,Y) - H(Y)\,. \end{aligned} \tag{1.31}$$

It is the measure of the average uncertainty of X if the value of Y is given. There is another intuitive way to arrive at the Eq. 1.31. In the earlier subsection we saw that joint entropy of the random variables is given by $H(X,Y)$ and knowing Y means we have acquired $H(Y)$ bits of information. Hence the amount of uncertainity in X given we know Y is $H(X|Y) = H(X,Y) - H(Y)$.

1.5.1.3 Basic properties of Shannon entropy

In the next part of this section we will discuss the quantum information theory, but, before that we will review and discuss some basic properties of Shannon entropy [2].

1. H(X,Y)=H(Y,X) and H(X:Y)=H(Y:X)

[1] Unless otherwise specified $\log x$ implies $\log_2 x$.

2. H(X|Y) $\geq$ 0 and thus H(X:Y) $\leq$ H(X), with equality iff X is a function of Y, i.e., X=f(Y)

3. H(X) $\leq$ H(X,Y), with equality iff Y=f(X).

4. H(X,Y) $\leq$ H(X) + H(Y) with equality iff X and Y are independent random variables. This property is known as subadditivity.

5. H(Y|X) $\leq$ H(Y) and thus H(X:Y) $\geq$ 0, with equality in each iff X and Y are independent random variables.

6. H(X,Y,Z) + H(Y) $\leq$ H(X,Y) + H(Y,Z), with equality iff Z $\rightarrow$ Y $\rightarrow$ X forms a Markov chain. This property is known as strong subadditivity.

7. H(X|Y,Z) $\leq$ H(X|Y).

1.5.2 Quantum Information Theory

Quantum information theory attempts to quantify the amount of uncertainty present in quantum systems. In section 1.2.3 we discussed how there are two kinds of uncertainty present in a quantum state, first due to lack of the knowledge of the ensemble and second due to the intrinsic randomness present in the quantum system. Analogous to Shannon entropy in classical theory we have von Neumann entropy to quantify this uncertainty in quantum information theory.

1.5.2.1 Von Neumann Entropy

Generalizing Shannon entropy to quantum states we get ***von Neumann entropy*** [2], defined as

$$S(\rho) = -\operatorname{tr}(\rho \log \rho) = -\sum_i \lambda_i \log \lambda_i, \tag{1.32}$$

where $\{\lambda_i\}$ is the set of eigen values of ρ. The von Neumann entropy for a pure state is always zero, as the state is completely known, and non-zero for statistical mixtures.

In similar fashion, joint entropy of a composite system ρ_{AB} is given by

$$S(\rho_{AB}) = -\operatorname{tr}(\rho_{AB} \log \rho_{AB}) = -\sum_i \mu_i \log \mu_i. \tag{1.33}$$

Here $\{\mu_i\}$ is the set of eigen values of ρ_{AB}.

1.5.2.2 Conditional Entropy

The Conditional Entropy for subsystem A given B in a composite quantum system ρ_{AB} is defined as,

$$\mathcal{S}(\rho_{A|B}) = \mathcal{S}(\rho_{AB}) - \mathcal{S}(\rho_B). \tag{1.34}$$

It is noteworthy that the conditional entropy can be negative when considering some entangled quantum systems. For example, consider a Bell state, $|\psi\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$, the joint entropy $\mathcal{S}(\rho_{AB}) = 0$ as it is a pure state, whereas the entropy is 1 for any one of the individual subsystems as it is maximally mixed state, $\rho_A = \rho_B = I/2$. Here I is the identity operator in $\mathbb{C}^2$. This implies the conditional entropy is $\mathcal{S}(\rho_{A|B}) = -1$.

1.6 Entanglement

Entanglement can be understood as an extraordinary degree of correlation between two or more quantum systems. Barring a few entangled states whose correlations can be reproduced by classical HVT models, there are some entangled states whose correlations cannot be reproduced classically. These entangled states play a key role in many interesting applications of quantum computation and quantum information. Algebraically, an entangled state is defined to be a quantum state that cannot be expressed as a tensor product of its subsystems. A famous example would be the Bell state

$$|\psi_{Bell}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{1.35}$$

Simply by using the ***proof by contradiction*** technique, one can see that $|\psi_{Bell}\rangle$ is entangled. The proof is as follows: Let us assume that $|\psi_{Bell}\rangle$ is separable, then

$$|\psi_{Bell}\rangle = (\alpha|0\rangle + \beta|1\rangle)(\gamma|0\rangle + \delta|1\rangle) = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

That implies, $\alpha\gamma = \frac{1}{\sqrt{2}}$, $\beta\delta = \frac{1}{\sqrt{2}}$, $\beta\gamma = 0$ and $\alpha\delta = 0$. That is a contradiction as the value of $(\alpha\gamma\beta\delta)$ cannot be both $\frac{1}{2}$ and 0. Similar proof technique can be used to check if multi-qubit pure states are entangled or not. However, it is not an easy problem to check whether a state is entangled or separable when we deal with mixed states, determining whether or not a given mixed state is separable is NP-hard [9]. A more generic definition of an entangled state (which includes both mixed and pure) is as follows, A quantum state ρ is entangled, iff it cannot be expressed as convex combination of tensor product of its subsystems as shown in Eq. 1.36

$$\rho \neq \sum_i p_i(\rho_i^A \otimes \rho_i^B \otimes \rho_i^C \otimes ...), \tag{1.36}$$

where each $p_i \geq 0$ and $\sum_i p_i = 1$. Also, if a state is not entangled then it is separable.

The importance of entanglement in many information theoretic tasks like teleportation and superdense coding and in quantum computation [2], makes it an important resource to be studied and understood. A lot of research has been done in detection and creation of entanglement. For bipartite systems of $2\otimes2$ and $2\otimes3$ dimensions a necessary and sufficient condition, known as Peres-Horodecki criterion has been given to distinguish entangled states from separable states [10, 11]. In multi qubit systems concepts like *task oriented entangled* states [12] have been introduced. As a consequence to that teleportation witnesses, thermodynamical witnesses [13] have been devised which can identify useful entangled states for various tasks. Generation of entanglement from separable states has been another topic of research. Unlike local unitaries, global unitaries can generate entanglement [14]. Using global unitaries one can take any pure state to any other pure state. However, global unitaries are not all that powerful when it comes to mixed states [15, 16, 17]. In the next chapter we will discuss the limits that mixed states impose on creation of entanglement, when done using global unitaries.

1.7 Conclusion

To summarize, we briefly discussed the historical developments that lead to the current quantum theory. We also discussed the basic mathematical formalism that is required for the study of this theory and which stands as preliminaries for understanding the second and the third chapters. Subsequently, we discussed the results on two of the very important milestones, namely the EPR paradox and the Bell's inequality. We also discussed the important results of classical and quantum information theory as they hold key significance to the work presented in the second chapter in relation to entanglement.

The following two chapters discuss upon a few limits that are posed to Quantum Information Theory. The second chapter examines some limits imposed due to the probability structure of mixed states. In mixed states the probability occurs at two levels, first it starts as a description of the system as a statistical mixture, second as the intrinsic superposition that is contained by each entity of the ensemble. We witness how the global unitaries, that can make any pure separable state entangled, fail to generate entanglement for a class of mixed states.

The third and final chapter discusses another limit that superposition imposes on quantum theory. That is, the impossibility to clone coherence of an arbitrary quantum state. It also addresses the logical implication of no-cloning of coherence which is the famous no-cloning theorem.

Chapter 2

Non-negativity of conditional von Neumann entropy and its absoluteness

"Correlations have physical reality; that which they correlate does not."

– N. David Mermin

In 1935, Einstein, Podolsky and Rosen first discussed the counterintuitive predictions of quantum mechanics about strongly correlated systems [18]. This strong correlation was eventually came to be termed as 'entanglement'. An entangled system is defined to be a quantum state that cannot be expressed as tensor product of it's subsystems. In an entangled system, one subsystem cannot be fully described without considering the other. This phenomenon lies at the heart of quantum mechanics. The phenonmenon of entanglement is not only of deep philosophical interest [5] but also is established as the most pivotal resource in various information processing tasks, like teleportation [19], superdense coding [20], key generation [21, 22], secret sharing [23], remote entanglement distribution [24] and many more [25, 26, 27]. However, not all entangled states can be directly used for an information processing task, pertinent mentions in this regard are the bound entangled states [28]. These entangled states are available when we go beyond $2 \otimes 2$ and $2 \otimes 3$ system, where we do not have necessary sufficient condition like Peres-Horodecki criterion [10, 11] for detection of entanglement. Some entangled states have to be processed by local filtering [29] before they can be used in a task. Consequently, teleportation witnesses, thermodynamical witnesses [13] have been devised which can identify useful entangled states for various tasks. In multi qubit systems concepts like *task oriented entangled* states [12] have been introduced.

The ubiquitous role of entanglement in information processing tasks has motivated recent research in the generation of entangled states from separable states. And, global unitary operations can play a significant role in this scenario as local unitaries cannot generate entanglement. Hence, in this chapter we will discuss the concept of *absoluteness*. The notion of *absoluteness* indicates that the state preserves a certain characteristic trait under global unitary transformations. *Absoluteness* was first discussed in the context of separability [15], which was later extended to define absolutely Bell-CHSH local states [17] and absolute unsteerability [16]. Having discussed absolutely separable and local class of states, we will analyse the absoluteness of conditional von Neumann entropy, another characteristic trait of quantum states, which unlike its classical counterpart can be negative. As negativity of conditional entropy is an important yardstick, our work in this chapter probes whether it is always possible to start with a state having non-negative conditional entropy and arrive at a state having negative conditional entropy via global unitaries. We show that there is a class of states which preserve the non-negativity of the conditional entropy under global unitary transformations. We call this class of states as Absolute Conditional von Neumann entropy Non Negative class (**ACVENN**). We characterize such states for $2 \otimes 2$ dimensional systems.

On a different perspective the characterization accentuates the detection of states whose conditional entropy becomes negative after the global unitary action. Interestingly, we show that this **ACVENN** class of states forms a set which is convex and compact. This feature enables the existence of hermitian witness operators. With these we can distinguish the unknown states which will have negative conditional entropy after the global unitary operation. We also show that this has immediate application in super dense coding and state merging as negativity of conditional entropy plays a key role in both these information processing tasks. Since separability and non-locality are also important distinctive features of quantum mechanics, we also deduce the connections of these states preserving the non-negativity of conditional entropy under global unitary with the absolutely separable states **AS** and the recently introduced absolutely Bell-CHSH local states **AL** [17].

The sections 2.1 and 2.2 deal with entanglement, separability, non-locality and their absoluteness. In the section 2.3 we give a general necessary and sufficient condition to characterize **ACVENN** class of states in the state space of two qubit systems. In section 2.4, we show that this **ACVENN** class of states is convex and compact which in principle allows to construct the witness operator for identifying those states which do not belong to this class. In section 2.5 we connect this **ACVENN** class of states with absolutely separable **AS** and absolutely local state **AL** states. In section 2.6 we show the potential application of charecterizing such states in various information processing tasks like super dense coding and state merging. Finally we conclude in section 2.7.

2.1 Entanglement, non-locality, conditional von Neumann entropy

In this section we will thoroughly discuss entanglement and its other characteristics like non-locality and negative conditional von Neumann entropy.

2.1.1 Entanglement versus separability

Entanglement is a quantum mechanical phenomena in which a quantum system of two or more subsystems have to be described with reference to each other, even though the individual subsystems are spatially separated.

Algebraically, a pure bipartite system $|\Psi\rangle_{AB}$ is said to be entangled iff it cannot be expressed as a tensor product of its subsystems as shown below.

$$|\Psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B.$$

In case of mixed bipartite systems a state is said to be entangled iff it cannot be expressed as a convex combination of tensor products of two subsystems, as shown in the equation below.

$$\rho_{AB} \neq \sum_i p_i \rho_i^A \otimes \rho_i^B, \sum_i p_i = 1.$$

If a state is not entangled, then it is said to be separable. Entanglement in quantum systems can be increased through various ways, local filtering, global unitaries, and through incoherent coherence operations on a non zero coherent subsystem [30].

Entanglement on one hand has various applications in quantum computing, quantum cryptography and many information theoretic tasks and on other hand has prompted some philosophical debates on quantum theory in the past. But, most of the debates that entanglement has prompted are to do more with the concept of non-locality and not entanglement per se. In the next section we will discuss how entanglement and non-locality are not one of the same.

2.1.2 Non-locality

It appears that entanglement forces measurement of one subsystem to instantaneously influence the outcome of the other correlated subsystems. This phenomena is best expressed through the concept of non-locality.

In this thesis we are referring to the Bell-CHSH locality. In section 1.4 of the previous chapter we discussed the derivation of Bell's inequality. We define Bell-CHSH local states as states that do not violate Bell's inequality. The states that do violate are termed as Bell-CHSH non-local states. It is noticed that Bell's theorem discriminates between quantum mechanics and all the other theories (hidden variable theories) where probablities of measurement arise due to the ignorance of the pre-existing local properties.

Analytically, a state ρ is said to be Bell-CHSH local iff

$$M(\rho) \leq 1.$$

Here $M(\rho) = \lambda_1 + \lambda_2$, where λ_1 and λ_2 are the maximum two eigen values of the matrix $T^{\dagger}T$ given that $T = [t_{ij}]$ is a correlation matrix of ρ, where $t_{ij} = Tr[\rho(\sigma_i \otimes \sigma_j)]$ with $[\sigma_i; i = \{1, 2, 3\}]$ are $2 \otimes 2$ Pauli matrices.

2.1.3 Conditional von Neumann entropy

Conditional von Neumann entropy just like entanglement and non-locality, is another characteristic trait of quantum states, that is because this value can be negative for some quantum systems unlike its classical counterpart. Conditional von Neumann entropy of system A given system B is denoted by,

$$S(\rho_{A|B}) = S(\rho_{AB}) - S(\rho_B),$$

where $S(\rho) = -tr(\rho ln \rho)$, denotes the von Neumann entropy of system ρ.

Following are some properties of von Neumann entropy [2].

1. $S(\rho) = 0$ iff ρ is a pure state.
2. The maximum von Neumann entropy is equal to $\ln(d)$ for a maximally mixed state, d being the dimension of the system.
3. $S(\rho)$ is invariant under unitary operations, i.e.,

   $$S(\rho) = S(U\rho U^{\dagger})$$

   where U is a unitary transformation.

4. $S(\rho)$ is additive for independent systems, i.e., given two density matrices ρ_A and ρ_B describing two independent systems A and B, we have

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B),$$

5. $S(\rho)$ is strongly subadditive for any three systems A, B, and C, i.e.,

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

   This automatically means that $S(\rho)$ is subadditive.

$$S(\rho_{AC}) \leq S(\rho_A) + S(\rho_C).$$

The reason for quickly going through the properties of von Neumann entropy will be clear in the upcoming sections as we use these properties to build the results presented in this chapter.

An operational interpretation of the quantum conditional entropy was provided in [31], in terms of state merging. Unlike its classical counterpart, this quantity can be negative [2], providing yet again a departure from classical information theory. The negativity of the conditional entropy also indicates the signature of entanglement, although the converse of the statement is not true as there are entangled states with non-negative conditional entropy. Conditional entropy also plays a key role in state merging [31] and dense coding [20] , as a bipartite quantum state is useful for dense coding in a sense that it will have quantum advantage if and only if it has a negative conditional entropy.

2.2 Absolutely separable, absolutely local states

In pure states, any state can be transformed to any other state using a global unitary, but, this result cannot be extended to all mixed states. It is very easy to picture this result for a single qubit system using a Bloch sphere. The unitaries for single qubit system are just rotational matrices about the state $\frac{\mathbb{I}}{2}$. These unitaries can take a state of some mixedness to another state of same mixedness but there does not exist any unitary that can increase or decrease mixedness of a state. Here, by mixedness of a state ρ we refer to the quantity $||\rho - \frac{\mathbb{I}}{2}||_{Frobenius}$ which is just a function of $tr(\rho^2)$.

Unlike Bloch sphere for a single qubit system, there is no geometric representation of $2 \otimes 2$ systems that we can further resort to. Therefore, in this section we will analytically discuss the absolutely separable and absolutely local class of states. In the later sections we characterize Absolute Conditional von Neumann entropy Non Negative class and establish its relation with **AS** and **AL** class.

2.2.1 Separable and Absolutely separable class of states

The generation of entanglement from separable states is one of the leading experimental frontiers at present [17]. When we go beyond the one qubit system to two qubit system, we come across the notion of entanglement, the states which can not be written as convex combination of tensor product of one qubit systems. The exact complement of this are those states for which composite system can be written as convex combination of tensor product of subsystems. However, the definition is not so straightforward when we go beyond two qubit pure states. For a mixed quantum system consisting of two subsystems the general definition of being separable is if its density matrix can be written as σ_{sep}=$\sum \lambda_i \sigma^A \otimes \sigma^B$, ($\sum \lambda_i = 1, \lambda_i \geq 0$), where σ^A and σ^B are density matrices for the two subsystems A and B [32]. The set **S** will denote the class of separable states. Lately, people have identified the class of absolutely separable states denoted by **AS** [15, 16] which are states that remain separable under all global unitary operations, i.e., $\mathbf{AS} = \{ \ \sigma_{as} : U\sigma_{as}U^\dagger$ is separable $\forall \ U\}$.

The problem of separability from spectrum was first handled in the case of $2 \otimes 2$ systems [33], where it was shown that σ is absolutely separable iff its eigenvalues (in descending order) satisfy $\lambda_1 \leq \lambda_3 + 2\sqrt{\lambda_2\lambda_4}$. As absolutely separable states remain separable under global unitary operations, such states cannot be used as input states for entanglement creation. Though pure product states are not absolutely separable, the same is not true for mixed separable states which become absolutely separable after crossing a given amount of mixedness [33]. Given the ubiquity of environmental interactions in turning pure states into mixed ones, it is of practical importance to determine whether a state is eligible to be used as input for entanglement generation. The usefulness of mixed separable states which are not absolutely separable was highlighted in [34] for the generation of maximally entangled mixed states. Mixed separable states from which entanglement can be created have also been studied in other works [35].

2.2.2 Local and Absolutely local class of states

We denote the set of all states which do not violate the Bell-CHSH inequality by **L** [36]. Recall that any density matrix in two qubits can be written in the canonical form, where T denotes the correlation matrix corresponding to ρ. The function $M(\rho)$ is defined as the sum of the maximum two eigenvalues of T^tT. Any state with $M(\rho) \leq 1$ is considered local with respect to the Bell-CHSH inequality [37]. Set of states that do not violate Bell-CHSH inequality is denoted by $\mathbf{L} = \{ \ \sigma_L : M(\sigma_L) \leq 1 \ \}$. Recently researchers were able to characterize the states which do not violate Bell-CHSH inequality under any global unitary. This set containing these states are denoted by **AL** [17] and is defined by **AL**= $\{\sigma_{al} : M(U\sigma_{al}U^\dagger) \leq 1 \ \forall U\}$.

A state $\rho \in \mathbf{AL}$ iff $(2\lambda_1 + 2\lambda_2 - 1)^2 + (2\lambda_1 + 2\lambda_3 - 1)^2 \leq 1$, where λ_1, λ_2 and λ_3 are the highest three eigen values of ρ in decreasing order.

Bell-CHSH non-locality [38, 39] which is more of a statistical feature exhibited by quantum states, had been first characterized in terms of the state parameters in [40]. Just like absolute separability, the relation between the violation Bell-CHSH inequality and the change of basis of the underlying Hilbert space which is brought about by global unitary operations is explored in the paper [41].

2.3 Characterization of Absolute Conditional von Neumann Entropy Non negative (ACVENN) class

In this section we will introduce the class of states for which the conditional von Neumann entropy remains non negative even after the application of global unitary operator. The characterization of these states enables us to identify states which can be made useful for some information processing task. The von Neumann entropy of a system ρ_{AB} with two subsystems *A* and *B* is denoted by $S(\rho_{AB})$. The conditional von Neumann entropy for ρ_{AB} entropy is defined as $S(\rho_{AB}) - S(\rho_A)$, where $S(\rho_A)$ denotes the von Neumann entropy of the subsystem *A*. We note the class of states for which the conditional von Neumann entropy is non negative. We denote this class by **CVENN** defined by **CVENN**$= \{\sigma_{cv} :$ S(σ_{cv}) - S$((\sigma_{cv})_A) \geq 0\}$.

ACVENN: The set of states whose conditional von Neumann entropy remains non-negative under any global unitary operations is denoted by **ACVENN**$= \{\sigma_{ac} :$ S$(U\sigma_{ac}U^\dagger)$ - S$[(U\sigma_{ac}U^\dagger)_A] \geq 0, \forall U\}$. The von Neumann entropy remains invariant under global unitary transformations, however the conditional entropy can change. We are interested in characterizing the set of states that preserves the non-negativity of the conditional entropy under unitary action on the composite system.

Theorem 1. *A state* $\sigma_{ac} \in \boldsymbol{ACVENN}$ *i* $S(\sigma_{ac}) \geq 1$.

Proof. Let $\sigma_{ac} \in$ **ACVENN**. Then, S$(U\sigma_{ac}U^\dagger)$ - S$[(U\sigma_{ac}U^\dagger)_A] \geq 0, \forall U$. This implies, S$(\sigma_{ac})$ - S$[(U\sigma_{ac}U^\dagger)_A] \geq 0, \forall U$, as von Neumann entropy is invariant under changes in the basis of σ_{ac}, i.e., S$(\sigma_{ac})=$ S$(U\sigma_{ac}U^\dagger)$ with U being any unitary transformation. Hence, we have S(σ_{ac}) $\geq$ S$[(U\sigma_{ac}U^\dagger)_A], \forall U$. The maximum value of S$[(U\sigma_{ac}U^\dagger)_A]$ is obtained at $(U\sigma_{ac}U^\dagger)_A = \frac{\mathbb{I}}{2}$ and the maximum value is 1. There always exists a unitary that converts the σ_{ac} to a Bell diagonal

σ_{bell} for a given spectrum. That is because,

$$\sigma_{ac} = \frac{1}{4}(I \otimes I + r.\sigma \otimes I + I \otimes s.\sigma + \sum_{n,m=1} t_{nm}\sigma_n \otimes \sigma_m).$$

Spectral decomposition of density matrices ensures that there will exist some basis $|e_1\rangle$, $|e_2\rangle$, $|e_3\rangle$, $|e_4\rangle$ in which σ_{ac} will be diagonalizable [2].

$$\sigma_{ac} = e_1 |e_1\rangle \langle e_1| + e_2 |e_2\rangle \langle e_2| + e_3 |e_3\rangle \langle e_3| + e_4 |e_4\rangle \langle e_4|,$$

and there exists some unitary transformation U, such that

$$|e_1\rangle = U |m_1\rangle, \quad |e_2\rangle = U |m_2\rangle,$$
$$|e_3\rangle = U |m_3\rangle, \quad |e_4\rangle = U |m_4\rangle,$$

where the $|m_1\rangle$, $|m_2\rangle$, $|m_3\rangle$, $|m_4\rangle$ are the Bell basis. Hence,

$$\begin{aligned} \sigma_{ac} &= e_1 U |m_1\rangle \langle m_1| U^\dagger + e_2 U |m_2\rangle \langle m_2| U^\dagger + \\ & e_3 U |m_3\rangle \langle m_3| U^\dagger + e_4 U |m_4\rangle \langle m_4| U^\dagger, \\ & = U(e_1 |m_1\rangle \langle m_1| + e_2 |m_2\rangle \langle m_2| \\ & + e_3 |m_3\rangle \langle m_3| + e_4 |m_4\rangle \langle m_4|)U^\dagger, \\ & = U\sigma_{bell}U^\dagger. \end{aligned} \tag{2.1}$$

We can see that σ_{ac} is a unitary transformation of some state σ_{bell}, where σ_{bell} is diagonalizable in Bell basis. And we know that for a Bell diagonal state the reduced subsystem $(\sigma_{bell})_A$ is $\frac{\mathbb{I}}{2}$. Therefore, S(σ_{ac}) $\geq$ S[$(U\sigma_{ac}U^\dagger)_A$], $\forall U \Rightarrow$ S(σ_{ac}) $\geq$ S[$(\sigma_{bell})_A$]$= S(\frac{\mathbb{I}}{2})= 1$.

Conversely let $S(\sigma_{ac}) \geq 1$, one can note that the maximum achievable von Neumann entropy of a subsystem is 1 in case of two qubit system. As under a unitary transformation, the entropy of the subsystem alone changes. Hence, for any state σ_{ac} whose von Neumann entropy is greater than equal to 1, we know that this state cannot have negative conditional entropy under any global unitary operations. Therefore, any state σ_{ac}, whose $S(\sigma_{ac}) \geq 1$ will $\in$ **ACVENN**. □

One may quickly note the following observations,

1. Any pure separable state has a non-negative conditional entropy and can be brought by some unitary to a maximally entangled state which now possesses a negative conditional

entropy and thus pure separable states can never belong to our desired class. Pure entangled states itself have a negative conditional entropy. Therefore, pure states are not eligible members of **ACVENN**.

2. The fact that some mixed states will be members of **ACVENN** is exemplified by the maximally mixed state which remains invariant under any global unitary operation and thus preserves the non-negativity of the conditional entropy. However, the maximally mixed state only constitutes a trivial example and we find that the class contains some very non-trivial states.

Example : A. Werner State

As an example, we first consider the example of Werner state. The density matrix representation of an Werner state is given by,

$$\sigma_{wer} = (1-p)(\mathbb{I}/4) + p|\psi\rangle\langle\psi|, \tag{2.2}$$

where, $|\psi\rangle = 1/\sqrt{2}(|00\rangle + |11\rangle)$ is the Bell state and p is the classical mixing parameter and $\mathbb{I}$ denotes identity.

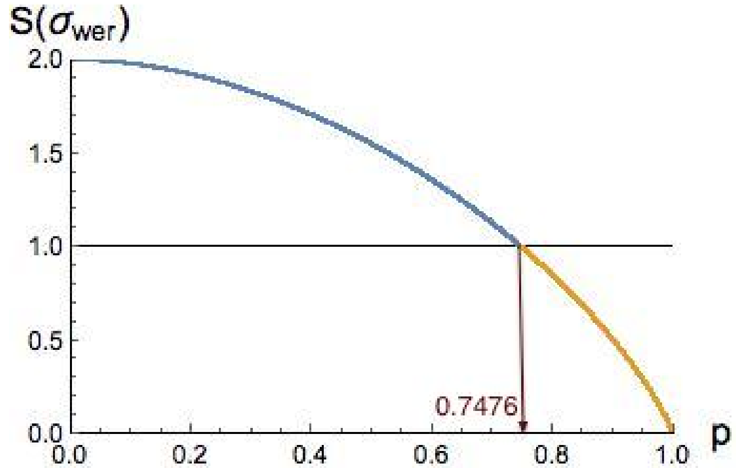

Figure 2.1: The von Neumann entropy of Werner state σ_{wer} against the classical mixing parameter p

In the figure 2.1 we have plotted the von Neumann entropy of the Werner state with respect to the mixing parameter p. Interestingly, we find that for all values of $p \in [0,\approx 0.7476]$, we have $S(\sigma_{wer}) \geq 1$. This clearly indicates the Werner state for values of $p \in [0,\approx 0.7476]$ falls within the **ACVENN** class.

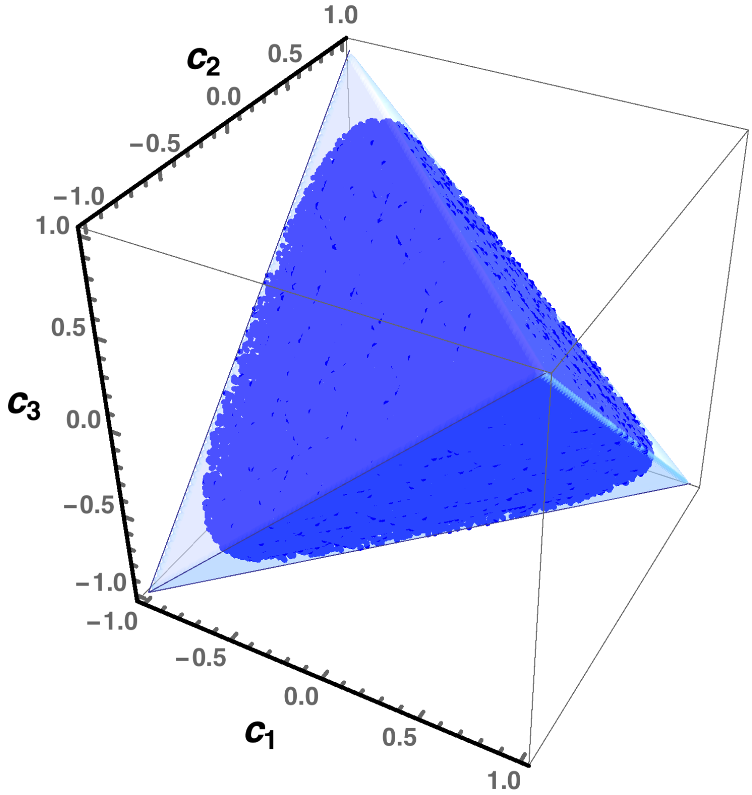

Figure 2.2: The von Neumann entropy of Bell diagonal state σ_{bell} against the parameter c_i

Example : B. Bell Diagonal States

Bell-diagonal states can be expressed as, $\sigma_{bell} = \{\vec{0}, \vec{0}, T^b\}$, where $\vec{0}$ is the Bloch vector which is a null vector and the correlation matrix is $T^b = (c_1, c_2, c_3)$ with $-1 \leqslant \mathrm{c}_i \leqslant 1$.

The eigenvalues λ_1, λ_2, λ_3, λ_4 of Bell diagonal states are expressed as, $\lambda_1 = \frac{1}{4}(\chi - 2c_1)$, $\lambda_2 = \frac{1}{4}(\chi - 2c_2)$, $\lambda_3 = \frac{1}{4}(\chi - 2c_3)$, $\lambda_4 = \frac{1}{4}(2 - \chi)$, where $\chi = 1 + c_1 + c_2 + c_3$. Therefore, necessary and sufficient condition for a Bell diagonal state to lie in **ACVENN** is given by $S(\sigma_{bell}) \geq 1$ which in terms of c_1, c_2, c_3 and χ becomes

$$\begin{aligned} \log((\chi-2c_2)(\chi-2c_3)(2-\chi)(\chi-2c_1)) \\ +c_1 \log\left(\frac{(\chi-2c_2)(\chi-2c_3)}{(2-\chi)(\chi-2c_1)}\right) \\ +c_2 \log\left(\frac{(\chi-2c_3)(\chi-2c_1)}{(\chi-2c_2)(2-\chi)}\right) \\ +c_3 \log\left(\frac{(\chi-2c_2)(\chi-2c_1)}{(\chi-2c_3)(2-\chi)}\right) \leq 4. \end{aligned} \tag{2.3}$$

In figure 2.2, we consider an exhaustive ensemble of 10^5 states within which the dark blue colour area at the centre of the octahedron determines the class of states for which $S(\sigma_{bell}) \geq 1$ and falls into our **ACVENN** class. The light blue areas at the corner are those areas whose conditional entropy can be made negative after the application of some global unitary transformation. It is evident from figure 2.2 that the non negativity of conditional entropy for most of the part of the Bell diagonal states remains invariant after the application of global unitary transformation.

2.4 ACVENN class and Existence of Witness

2.4.1 Witness operator and Geometric form of Hahn-Banach theorem

A geometric form of the Hahn-Banach theorem states that given a set that is convex and compact, there exists a hyperplane that can separate any point lying outside the set from the given set [1].

A witness operator W pertaining to a convex and compact set S will be a hermitian operator that satisfies the following conditions: (1) $Tr(W\sigma) \geq 0$, for all states $\sigma \in S$, (2) $Tr(W\chi) < 0$, for any state $\chi \notin S$ [13].

2.4.2 ACVENN is convex and compact

In this section we show that the **ACVENN** class which is a subset of the class **Q** is a convex and compact set. The class **Q** denotes the set of all two-qubit states. This helps us identifying the states whose conditional entropy remains negative even after the application of global unitary. We now present the proof that the set **ACVENN** is convex and compact.

Existence of Witness: The theorems below will support the existence of witness operators to classify **ACVENN** states from states which are not in **ACVENN**. See figure 2.3 for reference.

Theorem 2. *$ACVENN$ is convex.*

Proof. Consider $\sigma_1, \sigma_2 \in$ **ACVENN**. Therefore, $S(\sigma_i) \geq 1, i = 1, 2$. Now by the concavity of von Neumann entropy $S(\lambda\sigma_1 + (1-\lambda)\sigma_2) \geq 1$, where $\lambda \in [0, 1]$. Hence, $\lambda\sigma_1 + (1-\lambda)\sigma_2 \in$ **ACVENN** , implying **ACVENN** is convex. □

Theorem 3. *$ACVENN$ is compact subset of Q.*

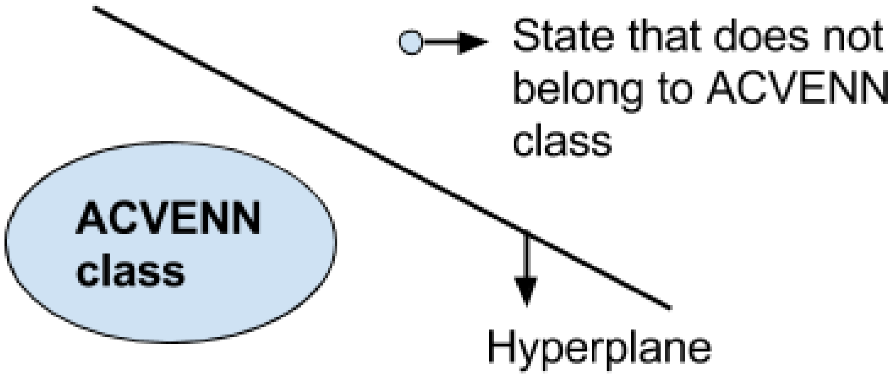

Figure 2.3: The set ACVENN is convex and compact, and using the Hahn-Banach theorem [1] it follows that any state not belonging to ACVENN can be separated from the states that belong to ACVENN by a hyperplane, thus providing for the existence of a witness.

Proof. Let us define a function $f : \mathbf{Q} \to \mathbb{R}$ as

$$f(\rho) = S(\rho), \tag{2.4}$$

as **ACVENN** = { σ_{ac} : S(σ_{ac})≥1 }, and f will have a maximum value of 2, we can say **ACVENN** = f^{-1}[1,2]. f is a continuous function as S is a continuous function [42]. Therefore, **ACVENN** = f^{-1}[1,2] is a closed set in **Q** defined under the trace norm. The set **ACVENN** is bounded as every density matrix has a bounded spectrum, i.e., their eigen values lies between 0 and 1. This proves that the **ACVENN** class is compact. □

The theorem now guarantees the existence of Hermitian operators to succesfully identify states that do not belong to **ACVENN**.

Next we estimate the size of the **ACVENN** class by taking the maximum and minimum distance from the identity ($\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$). The distance measure we have used in this context is the Frobenius norm which is given by $\|X\| = \sqrt{Tr(X^\dagger X)}$. Having already proved that **ACVENN** is a convex set, we try to find out the maximum and minimum distance from $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$.

For any general $\widetilde{\varrho}$, distance from $\frac{\mathbb{I}}{2} \otimes \frac{\mathbb{I}}{2}$ is given by $\|\widetilde{\varrho} - \frac{\mathbb{I}}{4}\| = \sqrt{Tr((\widetilde{\varrho} - \frac{\mathbb{I}}{4})^\dagger (\widetilde{\varrho} - \frac{\mathbb{I}}{4}))}$, which on solving further results to $\sqrt{Tr(\widetilde{\varrho}^2) - \frac{1}{4}}$.

To calculate the maximum distance we needed to maximise $\|\sigma - \frac{\mathbb{I}}{4}\|$, over all $\sigma \in$ **ACVENN**. Here we solve this problem numerically. After going through 2×10^5 **ACVENN** states the maximum distance we have is 0.645966 by the state whose eigen values were λ_1 = 0.809161, λ_2 = 0.0521141, λ_3 = 0.0595448, λ_4 = 0.0791805.

To calculate the minimum distance we needed to minimise $\|\rho - \frac{\mathbb{I}}{4}\|$, over all $\rho \notin$ **ACVENN**. Going through 1×10^5 **non-ACVENN** states numerically, we attained the minimum distance as 0.507225. This is given by a state whose eigen values were $\lambda_1 = 0.00014347$, $\lambda_2 = 0.000551157$, $\lambda_3 = 0.436523$, $\lambda_4 = 0.562783$.

In figure 2.4 we show rough estimation of the size of **ACVENN** class.

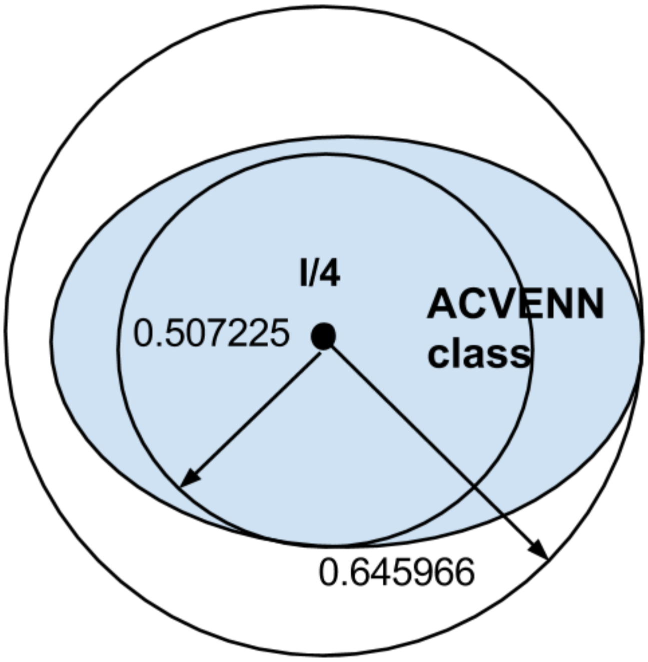

Figure 2.4: The figure depicts the approximate size of the **ACVENN** class

2.5 Relation between AS, ACVENN and AL

In this section we give a comparative picture of three classes of states. These classes remain invariant from the context of separability **AS**, non violation of Bell's inequlity **AL** and the non-negative conditional entropy **ACVENN** under global unitary transformation.

2.5.1 AS vs ACVENN

The figure 2.5 shows the relation between **AS** , **ACVENN** and separable states.

Lemma 1. ***AS*** $\subseteq$ ***ACVENN***

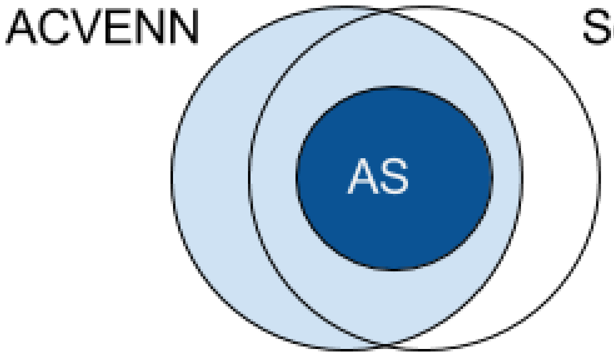

Figure 2.5: The figure depicts the relation between **ACVENN**, **AS** and separable class of states

Proof. Absolutely separable states preserve separability under any global unitary action. The non-negativity of conditional entropies is a necessary condition for separability [43]. All separable states have a non-negative conditional von Neumann entropy. Absolutely separable states(**AS**) remain separable under any unitary transformation. **AS** will always have non-negative conditional von Neumann entropy. So it will be a subset of **ACVENN** class. □

2.5.1.1 Illustration: A. Absolutely separable Werner states

Let us consider the Werner states σ_{wer} = p$|\psi\rangle\langle\psi|$ + $\frac{1-p}{4}\mathbb{I}$, where, $|\psi\rangle$ is the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. The state σ_{wer} belongs to **ACVENN** for values of p that satisfies the equation, $3(1-p)\log(1-p) + (1+3p)\log(1+3p) \leq 4$. Solving the inequality we get $p \in [0,\approx 0.7476]$ as obtained earlier. For the states to be in **AS**, one must have $a_1 \leq a_3 + 2\sqrt{a_2 a_4}$, where $a_1 = \frac{(1+3p)}{4}$, $a_2 = \frac{(1-p)}{4}$, $a_3 = \frac{(1-p)}{4}$, $a_4 = \frac{(1-p)}{4}$ are all the eigen values in descending order. σ_{wer} belongs to **AS** for values of p $\in [0, \frac{1}{3}]$. This gives an example of an absolutely separable state **AS** which is contained in **ACVENN** class.

2.5.1.2 Illustration: B. Incoherent in Computational Basis

Next, we give an example of a class of states which are incoherent in the computational basis,

$$\sigma_{comp} = a_1|00\rangle\langle 00 + a_2|01\rangle\langle 01| + a_3|10\rangle\langle 10| + a_4|11\rangle\langle 11|. \tag{2.5}$$

Taking the example of a state with eigenvalues $a_1 = \frac{5}{10}$, $a_2 = \frac{3}{10}$, $a_3 = \frac{2}{10}$, $a_4 = 0$, S(σ_{comp}) = -$\sum a_i \log_2 a_i \approx 1.485 \geq 1$. Hence, this state $\in$ **ACVENN**.

We see that $a_1 \geq a_2 \geq a_3 \geq a_4$. For the states to be in **AS**, one must have $a_1 \leq a_3 + 2\sqrt{a_2 a_4}$. In this specific case $a_1 = 0.5$, $a_3 + 2\sqrt{a_2 a_4} = 0.2$. Clearly, this shows that **AS** is a subset of **ACVENN** .

Theorem 4. ***AS*** $\subset$ ***ACVENN***

Proof. In lemma 1 it has been shown that **AS** $\subseteq$ **ACVENN**. In fact, we can say more than that. In view of the example on Werner states in Illustration 2.5.1.1, we have seen that there are states that donot belong to **AS** but belong to **ACVENN**. This shows that absolutely separable states (**AS**) form a proper subset of **ACVENN** □

After proving that **AS** $\subset$ **ACVENN** we want to estimate the minimum and maximum entropy recorded by the states belonging to **AS**. We solve this problem numerically as well. After going through 1×10^5 **AS** states the minimum entropy that we obtain is 1.58662. This is attained for a state with eigen values $\lambda_1 = 0.341023$, $\lambda_2 = 0.331417$, $\lambda_3 = 0.327411$, $\lambda_4 = 0.000148614$. We already know that the maximum entropy for **AS** is 2. This gives us a rough estimate of the volume of **AS** states lying within the **ACVENN** in terms of entropy.

2.5.2 AL vs ACVENN

The Werner states are absolutely local for the visibility factor $p \leq 1/\sqrt{2}$, and they belong to **ACVENN** for $p \leq 0.7476$. Therefore, the *absolutely Bell-CHSH local* Werner states form a subset of the **ACVENN** class. This is an interesting result as that would mean, there are states that violate *Bell-CHSH* inequality, and still under any unitary cannot be improved to a state with negative conditional entropy.

However, it is difficult to comment in general on the relation between **AL** and **ACVENN** class.

2.6 Applications: State Merging and Super Dense Coding

In this section we show how characterizing this **ACVENN** class of states helps to identify the states which are not useful for information processing tasks like super dense coding and state merging are made useful with the help of global unitary transformations. In either of these tasks we are able to detect a class of states which can be converted into super dense coding and state merging resource by applying global unitary transformations. In figure 2.6 we give pictorial description of two types of witness operators that can be created. W_{S_D} and W_{S_M} act as hyperplanes that detect states useful for superdense coding and state merging respectively from the states belonging to **ACVENN** class which can never be made useful for these information theoretic tasks by using global unitary operations.

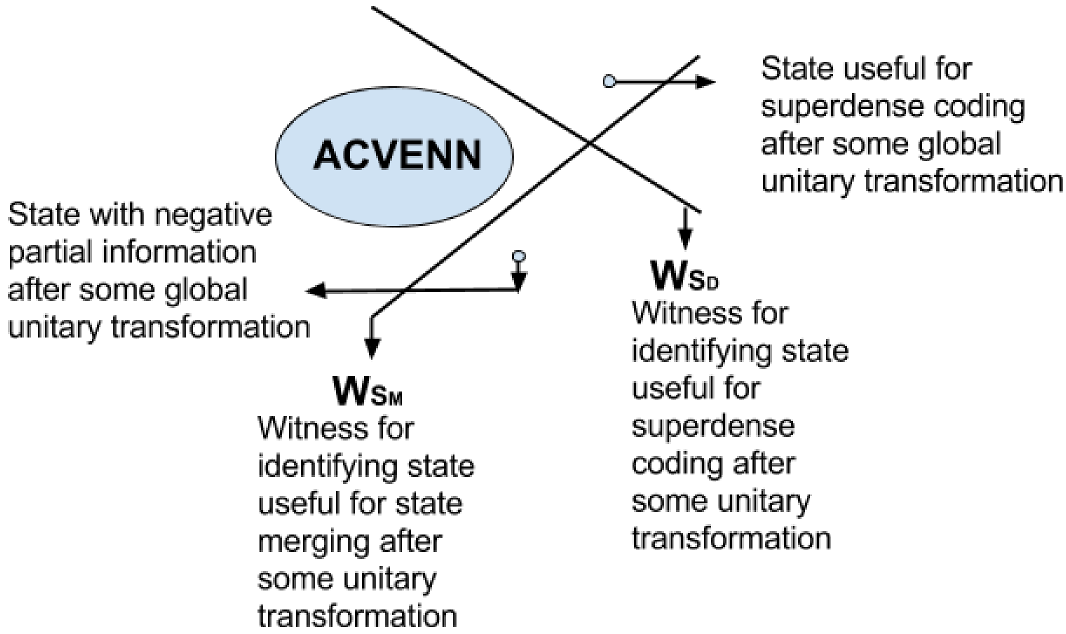

Figure 2.6: The role of witness in detecting the states which are useful for super dense coding and state merging after global unitary transformation

2.6.1 Super Dense coding

Quantum super dense coding involves in sending of classical information from one sender to the receiver when they are sharing a quantum resource in the form of an entangled state. More specifically, superdense coding is a technique used in quantum information theory to transmit classical information by sending quantum systems. It is quite well known that if we have a maximally entangled state in $H_d \otimes H_d$ as our resource, then we can send $2 \log d$ bits of classical information. In the asymptotic case, we know one can send $\log d + S(\rho)$ amount of bits. It had been seen that the number of classical bits one can transmit using a non-maximally entangled state in $H_d \otimes H_d$ as a resource is $(1 + p_0 \frac{d}{d-1}) \log d$, where p_0 is the smallest Schmidt coefficient. However, when the state is maximally entangled in its subspace then one can send up to $2 \log(d - 1)$ bits [20, 44].

In particular, super dense coding capacity for a mixed state ρ_{AB} in $D(H_d \otimes H_d)$ is defined by

$$\mathcal{C}_{AB} = max\{\log_2 d, \log_2 d + S(\rho_B) - S(\rho_{AB})\}, \tag{2.6}$$

where, $\rho_B = tr_A[\rho_{AB}]$ [20, 44]. $\mathcal{C}_{AB}$ is nothing but the amount of classical information that can be sent from system A to system B. Here we note that the expression $S(\rho_B) - S(\rho_{AB})$ can either be positive or negative. If it is positive then one can use the shared state to transfer bits greater than the classical limit of $\log_2 d$ bits. This in particular known as the quantum advantage where we can do more than the classical limit. For pure states, $S(\rho_{AB}) = 0$, then the super dense coding capacity is given by,

$$\mathcal{C}_{AB} = \log_2 d + S(\rho_B) = \log_2 d + E(\rho_{AB}), \tag{2.7}$$

where, the entanglement entropy $E(\rho_{AB})$ of a pure state ρ_{AB} is nothing but the von Neumann entropy $S(\rho_B)$ of the reduced subsystem ρ_B. The capacity will be maximum for the Bell states as $S(\rho_B)$ will be equal to 1. In a nutshell a state ρ_{AB} for which this expression $S(\rho_B) - S(\rho_{AB})$ is positive will give us a quantum advantage for superdense coding. In other words, a state with a negative conditional entropy $S(A|B)$ will be useful. It is obvious that not all states will have negative conditional entropy. The next important question is if we apply global unitary operator can we make a state which is not useful for super dense coding to a useful resource. In other words whether we can change the conditional entropy of the state from positive to negative. The answer is yes, however there will be some states for which we can not do that. These set of invariant states are nothing but previously described **ACVENN** class of states which can never be useful from the perspective of super dense coding, even after the application of global unitary operators. As we have seen previously that this class of state is convex and compact, then in principle it will be possible to create a witness operator (W_{S_D} as seen in figure 2.6) to detect the states which are initially not useful but made useful for superdense coding. It is important to mention here that this witness operator is not an witness operator to detect the states which are useful for super dense coding as opposed to the non useful state. Class of states useful for super dense coding is not a convex and compact set, so is the class of states that are not useful for super dense coding. This witness operator detects those states which need not be useful initially but can be made useful after global unitary transformation. Further we provide example to show all these kind of states.

2.6.1.1 Illustrations

For our first example let's consider a mixed separable state in $D(H_2 \otimes H_2)$ given by [13]

$$\rho = \begin{vmatrix} a & 0 & b & 0 \\ 0 & 0 & 0 & 0 \\ b & 0 & 1-a & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix}. \tag{2.8}$$

The eigen values for this state are $\frac{1-q}{2}$, $\frac{1+q}{2}$, 0, 0 and eigen values of the subsystem A are $\frac{1-q}{2}$, $\frac{1+q}{2}$ where $q = \sqrt{1 - 4a + 4a^2 + 4b^2}$. The state $\rho \in$ **ACVENN** iff $S(\rho) \geq 1$. In the current scenario that occurs only when $q = 0$. However, for no real values a and b is $q = 0$. Therefore we know that for no real values of a and b does this state belong to **ACVENN**. Thus $S(\rho_{A|B})$=0 for all real values of a and b, which is clearly not having negative conditional entropy, therefore, providing no quantum advantage. But, this on application of the unitary operator,

$$U_1 = \frac{1}{\sqrt{2}} \begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -1 & 0 & 0 & 1 \end{vmatrix}, \tag{2.9}$$

becomes,

$$\rho^{'} = \begin{vmatrix} \frac{a}{2} & 0 & \frac{b}{\sqrt{2}} & \frac{-a}{2} \\ 0 & 0 & 0 & 0 \\ \frac{b}{\sqrt{2}} & 0 & 1-a & \frac{-b}{\sqrt{2}} \\ \frac{-a}{2} & 0 & \frac{-b}{\sqrt{2}} & \frac{a}{2} \end{vmatrix}. \tag{2.10}$$

While the eigen values of $\rho^{'}$ remains unchanged, the eigen values of the subsystem $\rho^{'}_A$ becomes $\frac{1-q^{'}}{2}$, $\frac{1+q^{'}}{2}$ where $q^{'} = \sqrt{1-2a+a^2+2b^2}$. For all the values of a and b, where $q > q^{'}$ the state can be made useful for superdense coding. One such example would be when $a = 0.5$ and $b = 0.4$. Thus, we provide an example of a state which was not useful for super dense coding initially but after a unitary transformation it was made useful for super dense coding.

2.6.2 State merging and Partial Quantum Information

Another important information processing task is state merging. In the classical setting, the idea of state merging is essentially the following: Consider two parties Alice and Bob, where Bob has some prior information B and Alice has some missing information A (where A and B are random variables). At this point one important question is: If Bob wants to learn about A, how much additional information Alice does need to send him? It has been shown that only $H(A|B)$ bits suffices. In the quantum setting, Alice and Bob each possess a system in some unknown quantum state with joint density operator ρ_{AB}. Assuming that Bob is correlated with Alice, one asks how much additional quantum information Alice needs to send him, so that he has knowledge about the entire state. The amount of partial quantum information [31] that Alice needs to send Bob is given by the quantum conditional entropy, $S(A|B) = S(\rho_{AB}) - S(\rho_B)$.

In the information processing scenarios, it is important to ask this question. This communication measures the *partial information* one system needs conditioned on it's prior information. It is interesting to note that in principle this entropy can be positive ($S(A|B) > 0$), negative

($S(A|B) < 0$) and zero ($S(A|B) = 0$), where each have different meaning in the context of state merging. If the partial information is positive, its sender needs to communicate this number of quantum bits to the receiver; if zero it tells there is no need of such communication; if it is negative, the sender and receiver instead gain the corresponding potential for future quantum communication. So given a quantum state ρ_{AB}, shared between A and B, three possible cases arise, and we characterize the state based on these cases, namely for states with $S(A|B) > 0$ we denote them $\rho_{S(A|B)>0}$ and similarly other states as $\rho_{S(A|B)<0}$, $\rho_{S(A|B)=0}$. It is always useful from the information theoretic point of view to look out for the states $\rho_{S(A|B)<0}$ as they have the potential for future communication. It is needless to mention that not all states will be of this type. So the next question that becomes important in this context of global unitary is to find out states which are initially of the type $\rho_{S(A|B)>0}$ but can be converted to the type $\rho_{S(A|B)<0}$ after global unitary operations. Those states for which the conditional entropy remains positive even after all possible global unitary operations is nothing but the previously defined **ACVENN** class. Since we have already proved that **ACVENN** class is always convex and compact, this means that we can detect the states whose partial information can be made negative after the global unitary operation with the help of witness operator (W_{S_M} as seen in figure 2.6). Like in the case of super dense coding it is important to mention here also, that we are not showing that the set $\rho_{S(A|B)>0}$ is convex and compact but the set for which partial information remains positive after the global unitary operation (say, $U\rho_{S(A|B)}U^{\dagger} > 0$) is convex and compact. As a result we are not detecting the state for which the partial information is negative instead those states whose partial information can be made negative (say, $U\rho_{S(A|B)}U^{\dagger} < 0$) after the application of global unitary. We give examples to identify all these classes.

2.6.2.1 Illustrations

Lets take a mixed two qubit state,

$$\rho_{AB} = \frac{3}{4}|00\rangle\langle 00| + \frac{1}{4}|11\rangle\langle 11|, \tag{2.11}$$

we see, $S(\rho_{B|A})$=0. After the application of a unitary transformation U_2=U_1^{-1}, where U_1 is defined in equation 2.9 above. The state ρ_{AB} transforms to,

$$\rho^{'}_{AB} = \frac{1}{2}|00\rangle\langle 00| + \frac{1}{4}|00\rangle\langle 11| + \frac{1}{4}|11\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|. \tag{2.12}$$

The $S(\rho^{'}_{B|A})$ is -0.1887. Thus we give an example of a state which has non negative conditional entropy initially can be made negative with the help of a unitary transformation and also in principle one can construct the witness operator to detect such kind of states.

In this subsection we also ask this question: *For a given spectrum of density matrix for which states the minimum state merging cost will be achieved?*

Theorem 5. *For a given spectrum of density matrix the minimum state merging cost will be achieved at the Bell diagonal states .*

Proof. We know that the conditional entropy for the quantum state ρ_{AB} is given by the difference, $S(B|A) = S(\rho_{AB}) - S(\rho_A)$. Let us assume that the spectrum ρ_{AB} is fixed with eigen values a_i , $i = 1, 2, 3, 4$. Since the spectrum is fixed we have freedom to apply the global unitary operator. Now the question is to minimize $S(B|A)$, by using only global unitary operations. Since the global unitary operations will not change $S(\rho_{AB})$, we need to maximize $S(\rho_A)$. Now $S(\rho_A)$ is maximized if $\rho_A = \mathbb{I}/2$. The reduced density matrices for Bell diagonal state is $\mathbb{I}/2$. Therefore if one reaches Bell diagonal state by some global unitary no further maximization of $S(\rho_A)$ is possible. Thus for a given spectrum of density matrices the minimal merging cost is attained at the Bell-diagonal state. □

2.7 Conclusion

In this chapter we discussed entanglement and its characteristics like non-locality and conditional von Neumann entropy. Having discussed some of the characteristics, the *absoluteness* of these traits were also addressed. For a general two qubit system we are able to characterize class of states **ACVENN** whose von Neumann entropy will remain positive even after the application of global unitary operator. More specifically, we are able to show that the class of states with von Neumann entropy greater than 1, is the same **ACVENN** class of states.

We also found that this class of states is convex and compact, which guarantees the existence of witness for detecting the states which could have initially positive conditional entropy but have negative conditional entropy after the application of unitary operator. This in turn provides the technique to identify the states which are not initially useful but can be made useful in the information processing tasks like superdense coding and state merging.

Having discussed some limitations that the probability structure of mixed states pose to quantum information theory in this chapter, we now move to a different limit that superposition poses.

Chapter 3

Quantum coherence and its impossibility to be cloned

> "... what is proved by impossibility proofs is lack of imagination."
>
> – *John S. Bell*

The theory of wave-particle duality of matter dates back to as early as 1801, when Thomas Young conducted the famous double slit experiment and noticed that light behaved like waves. In the said experiment, Young throws a beam of light onto an opaque barrier with two extremely tiny slits, and a screen behind the barrier to capture the remenant light that passed through the holes. If light had behaved like corpuscles, as Isaac Newton had predicted in late seventeenth century, one would expect to see two bright dots on the screen, but, Young had an unexpected observation. He witnessed a complex interference pattern, a pattern that could only be produced if light were to behave like waves, and those waves having passed through the two holes had interferred. The experiment is regarded as one of most important discoveries to pave the way to the wave-particle theory of light. The theory of wave-particle duality was further strengthened in 1924 when Louis de Broglie proposed that matter too behaved like waves. In 1927, Clinton

Davisson and Lester Germer following Walter Elsasser's suggestion showed experimentally that infact, electrons did behaved like waves.

Eversince, the rise of quantum mechanics as a unified theory of waves and particles has strengthened the prominent role of coherence in physics. Coherence is said to be the fundamental signature of non-classicality: the wave-like nature of a quantum system. Inorder to measure the coherence of a system, we attempt to measure the amount of superposition that is present in that system. Superposition is the capacity that endows quantum states with the ability to generate true random numbers, realize parallel computing, and other tasks like quantum Bernouli factory. It has also been shown that coherence is the fundamental resource for creating entanglement [30] and magic [45]. Quantum superposition (and entanglement, a special case) plays a pivotal role in achieving information processing tasks that would otherwise be not possible by any classical resource. However, the interesting phenomenon is that, the same property constricts us to do certain tasks that are otherwise achievable classically.

It started with the no-cloning theorem which states that there does not exist any quantum operation which can perfectly duplicate a pure state [46]. In particular, the no-cloning theorem states that if we have cloning machine which can copy two orthogonal quantum states then with the same cloning machine it is impossible to create an identical copy of an arbitrary quantum state. However, this is not the only operation that quantum mechanics does not allow. Pati and Braunstein showed that we cannot delete either of the two quantum states perfectly [47]. In addition to these two famous no-cloning and no-deletion theorem there are many other no-go theorems like no-flipping (impossibility to flip an arbitrary quantum state) no-self replication (cannot have a universal quantum constructor) [48], no-partial erasure [49], no-splitting [50] and no- partial swapping [51] which together tells us about the indivisibility of the information content present in a quantum system.

The increasing importance of coherence as a resource and it's contribution towards making certain classically possible task impossible, makes it curiosly important that we study this area. Therefore, the chapter though starts with brief discussion on the resource theory of coherence and coherence measures, it more importantly discusses how coherence of an arbitrary quantum state can not be cloned. Yet another limit posed by this non-classical feature. The chapter further discusses examples of some cloners that can clone coherence perfectly for a class of states. We conclude this chapter by giving the implications of the no-cloning of coherence theorem.

3.1 Resource theory of coherence

A resource theory [52, 53] in general investigates the manipulation, detection, quantification and usage of a resource, first by identifying the free states that are created at no cost and then by identifying the free operations that can take one free state to another free state. As coherence adopts the quality of being a resource, a resource theory of coherence has also been developed [54]. The constraints that these resource theory pose is either due to some fundamental physical laws or due to the practical difficulty in executing some operations. Here, in this section we will briefly discuss the resource theory of coherence.

3.1.1 Incoherent quantum states

In general the first step to built a resource theory is to identify the free states for that resource. The free states are states that are created at zero cost. So in our case the free states would be states with zero coherence. As superposition of a state depends on the basis it is measured in, coherence invariably becomes a basis dependent quantity. By fixing a basis $\{|i\rangle\}$ usually known as ***reference basis***, we can say all density matrices that are diagonalizable in the reference basis are said to be ***incoherent states***, labelled by the set $\mathcal{I} \subset \mathcal{H}$. Hence, all density operators $\rho \in \mathcal{I}$ are of the form

$$\rho = \sum_{i=1}^{d} \alpha_i |i\rangle\langle i|,$$

where $\alpha_i \in \mathbb{R}, \forall i = 1..d$.

The Fig. 3.1 depicts all the incoherent states in $\{|0\rangle, |1\rangle\}$ basis.

3.1.2 Classes of Incoherent operations

Incoherent operations maps an incoherent state to another incoherent state. There are many classes of incoherent operations defined for resource theory of coherence. Here, we will briefly discuss the most important classes, their properties and their relation among each other.

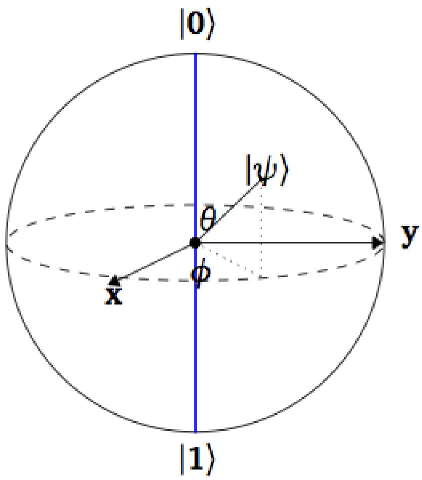

Figure 3.1: The blue line depicts all the incoherent states in $\{|0\rangle, |1\rangle\}$ basis

3.1.2.1 Maximally incoherent operations

Maximally incoherent operations (MIO), also known as incoherence preserving operations are those trace preserving completely positive, non-selective quantum operations $\Lambda : \mathcal{H} \to \mathcal{H}$ such that,

$$\Lambda[\mathcal{I}] \subset \mathcal{I}.$$

It should be noted that operations belonging to the class MIO cannot create coherence and these operations in general do not have a free dilation [55, 56].

3.1.2.2 Incoherent operations

Incoherent operations (IO) characterize a class of operations that are trace preserving completely positive maps $\Lambda : \mathcal{H} \to \mathcal{H}$, specified by a set of Kraus operators $\{\hat{K_n}\}$ satisfying $\sum_n \hat{K_n}^{\dagger}\hat{K_n} = 1$. The incoherent operators also fulfill $\hat{K_n}\mathcal{I}\hat{K_n}^{\dagger} \subset \mathcal{I}, \forall n$. Hence, incoherent operations are defined as those Kraus operations which map an incoherent state to another incoherent state and just like the MIO class this class of operations IO also does not admit free dilation [55, 56]. This definition of IO ensures that, in any of the possible outcomes of such an operation, coherence can never be generated from an incoherent input state, not even probabilistically.

3.1.2.3 Strictly incoherent operations

Strictly incoherent operations(SIO) [57] are the class of operations that are specified by a set of incoherent Kraus operators $\{\hat{K_n}\}$ such that the outcome of a measurement in the reference basis is independent of the coherence of the input, i.e.,

$$\langle i|K_n \rho K_n^\dagger|i\rangle = \langle i|K_n \Delta[\rho] K_n^\dagger|i\rangle, \forall n, \forall i,$$

where, $\Delta[\rho] = \sum_{i=0}^{d-1} |i\rangle\langle i|\rho|i\rangle\langle i|$ known as the dephasing operation.

3.1.2.4 Physical incoherent operations

Physical incoherent operations (PIO) are all operations that can be implemented by coupling the system to an environment in an incoherent state followed by a global incoherent unitary [55] unlike, the classes MIO, IO and SIO which do not have a free dilation.

MIO is the largest set of free operations allowed by a resource theory of coherence, the relation between MIO and the rest of the classes mentioned [55, 58] is as follows,

$$PIO \subset SIO \subset IO \subset MIO.$$

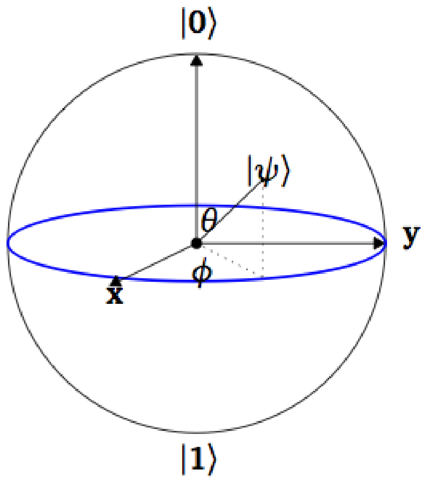

Figure 3.2: The blue circle depicts all the maximally coherent states in $\{|0\rangle, |1\rangle\}$ basis

3.1.3 Maximally coherent states

A d-dimensional maximally coherent state in the reference basis is given by,

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle.$$

Maximally coherent state in resource theory of coherence is analogous to maximally entangled state in resource theory of entanglement. Using the operations defined in the class IO, one can see that any d-dimensional state ρ can be created from the state $|\psi_d\rangle$ with certainty [54].

As the full set of maximally coherent states is obtained as the orbit of $|\psi_d\rangle$ under all incoherent unitaries [59], the Fig. 3.2 depicts all the maximally coherent states in $\{|0\rangle, |1\rangle\}$ basis.

3.1.4 Axioms of resource theory of coherence

Aberg in 2006 [60] presented the first axiomatic approach to quantify coherence, and later in 2014, Baumgratz ***et al.*** layed out an alternative framework for what qualifies to be a legitimate measure of coherence [54]. We will discuss the latter approach which is parrallel to the axiomatic quantification of entanglement [53].

1. *Nonnegativity:*
$$C(\rho) \geq 0,$$
and, $C(\rho) = 0$ iff $\rho \in \mathcal{I}$

2. *Monotonicity:* C does not increase under the action of incoherent operations, i.e.,
$$C(\Lambda[\rho]) \leq C(\rho),$$
for any incoherent operation Λ. While this condition was originally proposed for the class of operations IO [54], this condition can be easily extended to all the classes of incoherent operations we have mentioned earlier [61, 62, 63, 64].

3. *Strong Monotonicity:* On an average, C does not increase under selective measurements, that is
$$C(\rho) \geq \sum_n p_n C(\rho_n),$$
for all $\{\hat{K_n}\}$ with $\sum_n \hat{K_n}^\dagger \hat{K_n} = 1$ and $\hat{K_n} \mathcal{I} \hat{K_n}^\dagger \subset \mathcal{I}, \forall n$.

4. *Convexity:* Non-increasing under mixing of quantum states, that is
$$\sum_n p_n C(\rho_n) \geq C(\sum_n p_n \rho_n),$$
for any set of $\{\rho_n\}$ and any $p_n \geq 0$ with $\sum_n p_n = 1$

Inorder to be a meaningful resource for some coherence based task, the conditions 1 and 2 are seen to be the minimal requirements for any coherence measure C. Condition 3 states that coherence should not increase under incoherent measurements even if one has the knowledge of each individual measurement outcome. This condition 3, when combined with convexity 4, implies monotonicity 2 [54]. Usually, it is required that a coherence measure fulfills the conditions 1-4, unlike in some cases, where another additional two conditions are required to be satified [61, 62, 63, 64].

5. *Uniqueness* for pure states: For any pure state $|\psi\rangle$, *C* takes the form,

   $$C(|\psi\rangle\langle\psi|) = S(\Delta[|\psi\rangle\langle\psi|]),$$

   where, $S(\rho) = -tr[\rho log_2\rho]$ denotes the von Neumann entropy of ρ and $\Delta[\rho] = \sum_{i=0}^{d-1} |i\rangle\langle i|\rho|i\rangle\langle i|$ denotes the dephasing operation.

6. *Additivity:* C is additive under tensor products, i.e.,

   $$C(\rho \otimes \sigma) = C(\rho) + C(\sigma).$$

3.2 Quantum coherence measures

With the advent of coherence in quantum information science, a variety of coherence measures have been developed. In this section, we will briefly discuss some coherence measures. Some of them satisfy the axioms 1-4 from the section 3.1.4 and the others partially satisfy these axioms.

Coherence measures can be characterized into various types. *Distance-based coherence measures* facilitate the geometric approaches that are used to treat a huge class of problems such as the characterization and quantification of various quantum features. Operational measures such as *Entanglement-based coherence measure*, developed by Streltsov *et al.* in 2015, establishes that entanglement can be created from a state using incoherent operations if and only if the state had non-zero coherence. *Generalised coherence measures* have physical relevant and have potential application under specific contexts.

In this section, we will discuss various measures of coherence and illustrate the states with same coherence for some of these measures.

3.2.1 l_1-norm of coherence

This distance-based coherence measure is a valid coherence measure seeks to quantify the amount of linear superposition a quantum state possesses with respect to the reference basis. Given a state ρ, with its matrix elements as ρ_{ij}, the amount of coherence present in the state in the basis $\{|i\rangle\}$ is given by,

$$C_{l_1}(\rho) = min_{\delta \in \mathcal{I}} ||\rho - \delta||_{l_1} = \sum_{i \neq j} |\langle i|\rho|j\rangle|.$$

Since the l_1-norm is a function of the off-diagonal elements in a given density matrix, clearly the value of coherence will be zero in the eigen basis of the density matrix, where there are no off-diagonal elements.

If one choses the reference basis to be the eigen basis of σ_z then, for a single qubit state $\rho = \frac{I + \vec{m}.\vec{\sigma}}{2}$ with $\vec{m} = (m_x, m_y, m_z)$ as the Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices. Then coherence is given by $C_{l_1}(\rho) = \sqrt{m_x^2 + m_y^2}$. Hence, coherence only depends on m_x and m_y values. As shown in the Fig. 3.3, all the states that lie on the curved surface of the cylinder with radius $\sqrt{m_x^2 + m_y^2}$ will have the same coherence.

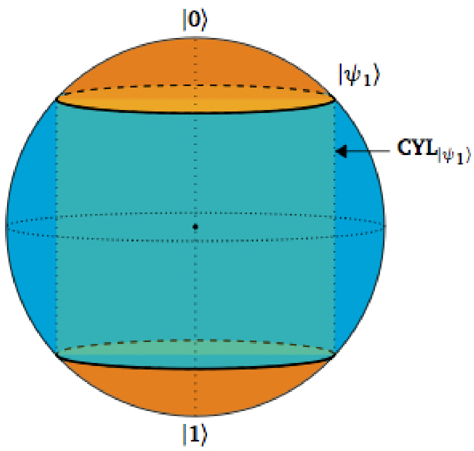

Figure 3.3: The hollow cylinder $CYL_{|\psi_1\rangle}$ represents all the states (pure as well as mixed) that have same coherence ($l_1 - norm$) in $\{|0\rangle, |1\rangle\}$ basis as state $|\psi_1\rangle$. The blue zone on the surface of the Bloch sphere denotes the pure states whose coherence is greater than of $|\psi_1\rangle$. The remaining zone shown in orange represent the pure states whose coherence is less than that of $|\psi_1\rangle$.

3.2.2 Relative entropy of coherence

The relative entropy of coherence another distance-based coherence measure is also a bonafide measure of coherence [54], defined as,

$$C_r(\rho) = min_{\sigma \in \mathcal{I}} S(\rho||\sigma) = S(\rho_{diag}) - S(\rho)$$

where, $S(\rho||\sigma)$ is the relative entropy between ρ and σ and $\rho_{diag} = \sum_i \rho_{ii}|i\rangle\langle i|$. This is an entropic measure of coherence which has a clear physical interpretation, as $C_r(\rho)$ equals to the optimal rate of the distilled maximally coherent states by incoherent operations in the asymptotic limit of many copies of ρ [57].

3.2.3 Modified trace distance measure of coherence

The modified trace distance measure of coherence is defined as,

$$C_{tr}(\rho) = min_{\lambda \geq 0, \sigma \in \mathcal{I}} ||\rho - \lambda\sigma||_{tr},$$

where $||A||_{tr} = Tr\sqrt{AA^\dagger}$.

Yang *et al.* [65] showed that for single qubit quantum systems the modified distance measure of coherence and the l_1-norm of coherence measures are equivalent, $C_{tr}(\rho) = C_{l_1}(\rho)$.

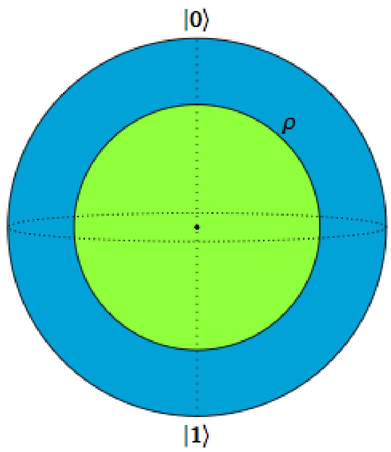

Figure 3.4: The surface of the green sphere in the Bloch sphere denotes all the mixed states that has same coherence ($Basis - independent\ coherence\ measure$) value to that of ρ

3.2.4 Basis-independent coherence measure

Basis-independent coherence measure is a generalized coherence measure that was developed by Yao *et al.* in 2016 [66].

$$C_F(\rho) = \sqrt{\frac{d}{d-1}}||\rho - \frac{I}{d}||_2.$$

The $C_F(\rho)$ can be analytically calculated as,

$$C_F(\rho) = \sqrt{\frac{d.tr(\rho^2) - 1}{d-1}} = \sqrt{\frac{d.I_{BZ}}{d-1}},$$

where I_{BZ} equals to the Brukner-Zeilinger invariant information thus endows $C_F(\rho)$ to have a clear physical meaning [67]. Moreover, it also provides loose lower bounds for the l_1-norm of coherence and trace norm of coherence [66], i.e.,

$$C_{l_1}(\rho) \leq \sqrt{d(d-1)}C_F(\rho),$$

$$C_{tr}(\rho) \leq \sqrt{d-1}C_F(\rho),$$

For a single qubit state $\rho = \frac{I+\vec{m}.\vec{\sigma}}{2}$, the basis-independent coherence is given by $C_F(\rho) = \sqrt{m_x^2 + m_y^2 + m_z^2}$ and all states with the same coherence are illustrated in the Fig. 3.4.

3.3 Impossibility of cloning of Quantum Coherence

In 1982, Wooters and Zurich presented a simple but elegant proof showing how a single quantum bit cannot be cloned. In this section we argue a stronger statement than that which suggests that the quantum coherence cannot be cloned. As no-cloning of coherence would imply no-cloning of state, we argue that quantumness of a single two-level quantum system is due to the coherence present in the system. The no-cloning theorem states the impossibility to clone an arbitrary quantum state. However, this inability does not lead directly to no-cloning of quantum coherence. Also, as coherence is an important resource for various information theoretic tasks it is imperative that we attempt to understand how to maintain the coherence of a single system. There are of course many ways to maintain this coherence, one particular way would be to clone the coherence. And before we attempt to do that it is important to see whether it is even possible to clone coherence.

One fundamental implication of this result is that quantum wave cannot be cloned. Since a quantum entity displays both the wave and particle nature, our result confirms that wave phenomenon in quantum world is different than the wave in the classical world. Further, we also discuss that cloning of coherence is a different problem from cloning of a state, which we already know is impossible.

Thus, the no-cloning of quantum coherence is more powerful than no-cloning of an arbitrary quantum state.

3.3.1 Cloning of coherence is different from cloning of state

In quantum mechanics the wave function describes the physical system completely. Hence, cloning of quantum states would mean cloning of both wave and particle aspects of the entity. However, when we clone coherence, we try to clone only the wave aspect. In this section, we show that indeed it is so and these two cloners are different. We illustrate some instances where coherence cloning is not the same as the cloning of quantum states. As an example, consider the Wootter-Zurich (WZ) cloning machine [46] that performs the following operation

$$\begin{aligned} |0\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |0\rangle_A|0\rangle_B|X_1\rangle_C, \\ |1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |1\rangle_A|1\rangle_B|X_2\rangle_C. \end{aligned} \tag{3.1}$$

Here A, B and C are the input, output and machine qubits respectively. When we apply the same W-Z cloning machine on an arbitrary quantum state $\alpha|0\rangle + \beta|1\rangle$ ($|\alpha|^2 + |\beta|^2 = 1$) whose coherence is $2|\alpha||\beta|$ in $\{|0\rangle, |1\rangle\}$ basis, it transforms to the state $\alpha^2|0\rangle\langle 0| + \beta^2|1\rangle\langle 1|$ which has zero coherence in $\{|0\rangle, |1\rangle\}$ basis. This shows that even if the state gets cloned approximately with W-Z cloning machine, there is no cloning of the coherence as W-Z cloning machine does not take into account the off-diagonal terms of the state.

Let us consider another example of the Buzek Hillery (BH) cloning machine [68] which is proved to be optimal state independent quantum cloning machine [69]. The 2-dimensional BH cloning transformation is given as

$$\begin{aligned} |\Psi_1\rangle_A|0\rangle_B|X_0\rangle_C \to\, & c\,|\Psi_1\rangle_A|\Psi_1\rangle_B|X_{11}\rangle_C \\ & + d\,(|\Psi_1\rangle_A|\Psi_2\rangle_B + |\Psi_2\rangle_A|\Psi_1\rangle_B)\,|Y_{12}\rangle_C\,, \\ |\Psi_2\rangle_A|0\rangle_B|X_0\rangle_C \to\, & c\,|\Psi_2\rangle_A|\Psi_2\rangle_B|X_{22}\rangle_C \\ & + d\,(|\Psi_2\rangle_A|\Psi_1\rangle_B + |\Psi_1\rangle_A|\Psi_2\rangle_B)\,|Y_{21}\rangle_C\,, \end{aligned}$$

where the coefficients c and d are real. The notation A, B and C represents the input, output and machine qubits respectively. In case of cloning a single qubit, using the no-signaling constraint and the fidelity as parameter of quantum cloning machine, Gisin proved that B-H state independent quantum cloner is the optimal one with the fidelity $\frac{5}{6}$ [69]. But if we consider the ratio of the final coherence to the initial coherence (l_1- norm), the B-H cloner gives $\frac{2}{3}$. That is, in other words, two thirds of coherence is getting copied with the B-H cloner. This example shows that even though information gets cloned upto $\frac{5}{6}$, coherence gets cloned only upto $\frac{2}{3}$. Possibly, this suggests us that when we clone quantum information we try to clone both the wave information and the particle information. As BH machine only clones wave information upto $\frac{2}{3}$ it may be the case that the higher value of $\frac{5}{6}$ is due to particle nature getting cloned more compared to wave information.

3.3.2 No-Cloning of Quantum Coherence

In this section we show that it is impossible to clone coherence of an arbitrary quantum state. We attempt to build the theorems without the constraints of any measure of coherence.

The proof starts with the assumption that it is possible to clone coherence perfectly for a pair of orthogonal states. The assumption is argued to be a valid assumption, because it is possible to have a cloner that clones two orthogonal states perfectly. If two orthogonal states can be cloned given a cloner, then definitely their coherence can be cloned.

3.3.2.1 Coherence of an arbitrary quantum state cannot be cloned with transformations using machine states

Let U_{cc} be the unitary transformation that produces two copies of coherence starting from two orthogonal quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$. The cloning transformation for coherence is given by,

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\Psi_1\rangle_{AB}|X_1\rangle_C, \\ |\psi_2\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\Psi_2\rangle_{AB}|X_2\rangle_C, \end{aligned} \tag{3.2}$$

where $|\psi_1\rangle$, $|\psi_2\rangle$ are the input states, $|0\rangle$ is the blank state and $|X_0\rangle$ is the initial machine state. Also, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ are states whose subsystems A and B have coherence same as that of the input states, and, $|X_1\rangle$ and $|X_2\rangle$ are the corresponding final machine states. The machine states satisfy $\langle X_2|X_1\rangle = 0$ due to unitarity of the transformation.

Let us represent the two orthogonal states $|\psi_1\rangle_A$ and $|\psi_2\rangle_A$ in the $\{|0\rangle, |1\rangle\}$ basis as, $|\psi_1\rangle_A = a|0\rangle_A + b|1\rangle_A$ and $|\psi_2\rangle_A = b^*|0\rangle_A - a^*|1\rangle_A$.

As the transformation demands coherence to be perfectly copied, we must have $C_l(|\psi_1\rangle_A) = C_l(\rho_A{}') = C_l(\rho_B{}')$ and $C_l(|\psi_2\rangle_A) = C_l(\rho_A{}'') = C_l(\rho_B{}'')$ where,

$$\begin{aligned} \rho_A{}' = Tr_B|\Psi_1\rangle_{AB\,AB}\langle\Psi_1|, \rho_B{}' = Tr_A|\Psi_1\rangle_{AB\,AB}\langle\Psi_1|, \\ \rho_A{}'' = Tr_B|\Psi_2\rangle_{AB\,AB}\langle\Psi_2|, \rho_B{}'' = Tr_A|\Psi_2\rangle_{AB\,AB}\langle\Psi_2|. \end{aligned} \tag{3.3}$$

Since the coherence of the orthogonal states for a 2-dimensional quantum system are same, we have $C(|\psi_1\rangle_A) = C(|\psi_2\rangle_A)$, where coherence of ρ is denoted by $C(\rho)$. It may be noted that in the case of cloning of quantum states we require two identical copies of the input state at the output port. However, for cloning of coherence this is not the case as there can be two non identical state with the same coherence.

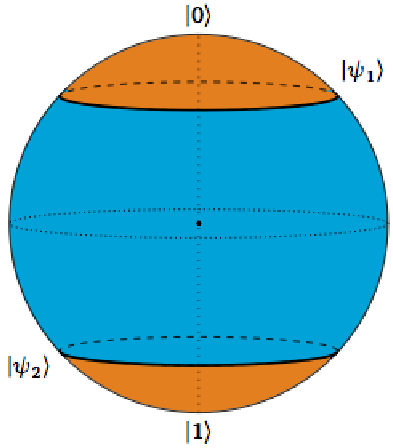

Figure 3.5: The curved surface on the blue region of the Bloch sphere denotes all the pure states whose coherence ($l_1 - norm$) can not be cloned, given the input states are $|\psi_1\rangle, |\psi_2\rangle$.

Theorem 6. *It is impossible to clone the coherence of an arbitrary quantum state* $|\psi\rangle$=$\alpha|\psi_1\rangle + \beta|\psi_2\rangle$*, with the cloning transformations given by equation (3.2) when the coherence of the state* $|\psi\rangle$ *is more than the coherence of the states* $|\psi_i\rangle$ $(i = 1, 2)$ *for a fixed choice of basis* $\{|0\rangle, |1\rangle\}$ *.*

Proof. Any arbitrary state in $|\psi_1\rangle_A$, $|\psi_2\rangle_A$ basis can be written as $|\psi\rangle_A = \alpha|\psi_1\rangle_A + \beta|\psi_2\rangle_A$. After the application of cloning transformation U_{cc}, the arbitrary state along with the blank and machine states becomes $(\alpha|\Psi_1\rangle_{AB}|X_1\rangle_C + \beta|\Psi_2\rangle_{AB}|X_2\rangle_C)$. Tracing out the subsystems B and C, we get $\rho_A^{final} = |\alpha|^2\rho_A{}' + |\beta|^2|\rho_A{}''$. From the convexity property of coherence measure [54] we have, $C(\rho_A^{final}) \leq (|\alpha|^2 C(\rho_A{}') + |\beta|^2 C(\rho_A{}'')) = C(|\psi_i\rangle), i = 1, 2$. Therefore, the final coherence of the subsystem A is at most $C(|\psi_i\rangle), i = 1, 2$. Therefore, it is evident that all the input states $|\psi\rangle$ whose initial coherence $C(|\psi\rangle)$ is greater than $C(|\psi_i\rangle), i = 1, 2$, which is the coherence of the known orthogonal states, it is impossible to clone the coherence perfectly. □

It is interesting to see that the impossibility of cloning the coherence of an arbitrary quantum state depends upon the choice of the known states. And as long as the known states $|\psi_i\rangle, i = 1, 2$ are not maximally coherent states, we are sure to find states $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$ where the $C(|\psi\rangle) > C(|\psi_i\rangle), i = 1, 2$.

So to summarize, the theorem only tells us that given a choice of known orthogonal states there does not exist any universal cloner which will clone all pure states. However, the theorem is only true as long as the orthogonal states are not from the equatorial circle of the Bloch sphere (as all the maximally coherent states lie on the equatorial circle of the Bloch sphere). In that scenario, we do not have any input state with a coherence greater than the coherence of these equatorial orthogonal states. The important question is whether for such choice of cloner it is possible to clone all the states on the surface of Bloch sphere. It is important to note that as the cloning transformation assume that the coherence of the input states are perfectly cloned, incase of the input states being the equatorial states, the cloning transformation will be as follows,

$$\begin{aligned} |\psi_{1(E)}\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_1{}'\rangle_A|\psi_1{}''\rangle_B|X_1\rangle_C \\ |\psi_{2(E)}\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_2{}'\rangle_A|\psi_2{}''\rangle_B|X_2\rangle_C \end{aligned} \tag{3.4}$$

because there are no mixed states that will be maximally coherent. $\langle X_2|X_1\rangle = 0$ due to unitarity of the transformation here as well.

Corollary 7. *For a cloning transformation given by equation (3.4), with the choice of known orthogonal states $|\psi_{1(E)}\rangle$ and $|\psi_{2(E)}\rangle$, taken from the equator, it is impossible to clone the coherence of an arbitrary quantum state $|\psi\rangle$=$\alpha|\psi_{1(E)}\rangle + \beta|\psi_{2(E)}\rangle$, for a fixed choice of basis $\{|0\rangle, |1\rangle\}$.*

Proof. Let $|\phi_{(E)}\rangle = \gamma|\psi_{1(E)}\rangle + \delta|\psi_{2(E)}\rangle$ be an arbitrary quantum state on the equatorial circle of the Bloch sphere. If there exists a universal cloner defined by the cloning transformations given in 3.4 that will clone coherence perfectly for any arbitrary state, then, the same cloner should also clone coherence of $|\phi_{(E)}\rangle$ perfectly. Let us say $\rho_{(E)final}$ be the transformed state under the tranformation defined in 3.4. $\rho_{(E)final} = |\gamma|^2|\psi_1^{'}\rangle\langle\psi_1{}'| + |\delta|^2|\psi_2^{'}\rangle\langle\psi_2{}'|$. If the tranformations in 3.4 clone coherence perfectly then $\rho_{(E)final}$ has to be a pure state because there are no mixed states that are maximally coherent. And the only possible way for $\rho_{(E)final}$ to be pure is $|\psi_1{}'\rangle_A = |\psi_2{}'\rangle_A$.

Arguing in the similar way for subsystem B we get $|\psi_1{}''\rangle_B = |\psi_2{}''\rangle_B$, then, any non-maximally coherent state $|\psi\rangle$ can be written as $\alpha|\psi_{1(E)}\rangle + \beta|\psi_{2(E)}\rangle$, but under this transformation rules the subsystem A will transform to $|\psi_1^{'}\rangle\langle\psi_1{}'|$ and subsystem B to $|\psi_1^{''}\rangle\langle\psi_1{}''|$. Consequently, the coherence of subsystem A and subsystem B becomes maximum. Hence, perfect cloning of coherence does not take place. □

3.3.2.2 Coherence of an arbitrary quantum state cannot be cloned with transformations without machine states

In the previous section, we have shown that there does not exist universal cloning transformation which will be able to clone the coherence of any arbitrary state. In the previous proof the cloning transformation includes the ancilla states representing the machine states. In this section, we investigate whether there exists any unitary in general which will act on the input state and blank state without invoking ancillary state that will clone coherence for any arbitrary state. We find that there exists no such unitary. Like in the previous section, here also we assume that the perfect cloning is possible for two known orthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$. The transformation is given by

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B &\longrightarrow |\Psi_1\rangle_{AB}, \\ |\psi_2\rangle_A|0\rangle_B &\longrightarrow |\Psi_2\rangle_{AB}, \end{aligned} \tag{3.5}$$

where, $\langle\psi_1|\psi_2\rangle = 0$. Therefore, $\langle\Psi_1|\Psi_2\rangle = 0$.

Theorem 8. *It is impossible to clone the coherence of any arbitrary quantum state $|\psi\rangle=\alpha|\psi_1\rangle+\beta|\psi_2\rangle$, with the cloning transformations given by Eq. 3.5.*

Proof. Let us assume that there exists a unitary that clones coherence of any arbitrary quantum state. Then this unitary should clone coherence for the states $|+\rangle$ and $|-\rangle$ as well. As these states are maximally coherent states, and this machine can clone the coherence perfectly, then the output states should also be maximally coherent states. The transformation would be given by Eq. 3.6. The output states in this case are all pure because there are no mixed states which are maximally coherent.

$$\begin{aligned} |+\rangle_A|0\rangle_B &\longrightarrow |\psi_1{}'\rangle_A|\psi_1{}''\rangle_B \\ |-\rangle_A|0\rangle_B &\longrightarrow |\psi_2{}'\rangle_A|\psi_2{}''\rangle_B \end{aligned} \tag{3.6}$$

where, either $\langle\psi_1{}'|\psi_2{}'\rangle = 0$ or $\langle\psi_1{}''|\psi_2{}''\rangle = 0$. Let $|\phi\rangle = \gamma|+\rangle + \delta|-\rangle$ be an arbitrary quantum state on the equatorial circle of the Bloch sphere. The transformation as given in Eq. 3.6 results in the state of the system AB to $|\Phi_{final}\rangle = \gamma|\psi_1{}'\rangle_A|\psi_1{}''\rangle_B + \delta|\psi_2{}'\rangle_A|\psi_2{}''\rangle_B$. For the coherence to be cloned perfectly, $|\Phi_{final}\rangle$ needs to be a separable system of two maximally coherent states. This makes either $|\psi_1{}'\rangle_A = |\psi_2{}'\rangle_A$ or $|\psi_1{}''\rangle_B = |\psi_2{}''\rangle_B$ because one of this pair has to be orthogonal.

Without loss of generality lets assume that $\langle\psi_1'|\psi_2'\rangle = 0$ and $|\psi_1''\rangle_B = |\psi_2''\rangle_B$ then any non-maximally coherent state $|\psi_1\rangle$ from Eq. 3.5 can be written as $\alpha|+\rangle + \beta|-\rangle$, but under this transformation rules the system will transform to $(\alpha|\psi_1'\rangle+\beta|\psi_2'\rangle)_A|\psi_1''\rangle_B$. Though the coherence of subsystem A is preserved the coherence of subsystem B is still maximum. □

3.4 Coherence-clonable states with some examples of coherence-cloners

In the previous section we discussed how there does not exist a universal cloner that will clone coherence for any arbitrary state. In this section we will look at a class of states whose coherence can be cloned given a cloner. Also note that, unlike in the previous section where we discussed results irrespective of any coherence measure, in this chapter we resort to the l_1-norm of coherence. And our examples of coherence cloners are based on the same measure. And, as we have seen that different coherence measure provides different ordering of states, we cannot extend the results in this section to other coherence measure.

3.4.1 Example of a cloner that can clone coherence for a maximum range of states

Let us take the two orthogonal input states $|\psi_1\rangle_A$ and $|\psi_2\rangle_A$ that lie on the equatorial circle of the Bloch sphere, i.e. $C_l(|\psi_1\rangle) = C_l(|\psi_2\rangle) = 1$. Here $C_l(\rho)$ denotes the l_1-norm coherence of ρ. The output states will have equal coherence to that of the input states. Then the cloning transformation is given by

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_1'\rangle_A|\psi_1''\rangle_B|X_1\rangle_C, \\ |\psi_2\rangle_A|0\rangle_B|X_0\rangle_C &\longrightarrow |\psi_2'\rangle_A|\psi_2''\rangle_B|X_2\rangle_C, \end{aligned} \tag{3.7}$$

where, $\langle\psi_1^{'}|\psi_2^{'}\rangle = 0$ and $\langle\psi_1^{''}|\psi_2^{''}\rangle = 0$.

Interestingly, it is observed that class of states whose coherence can be cloned perfectly given the transformations defined in Eq. 3.7 and the conditions $\langle\psi_1^{'}|\psi_2^{'}\rangle = 0$ and $\langle\psi_1^{''}|\psi_2^{''}\rangle = 0$ are the states that lie on the great circle passing through the states $|\psi_1\rangle$, $|\psi_2\rangle$, $|0\rangle$ and $|1\rangle$ on the Bloch sphere as seen in the Fig. 3.6.

The calculations are as follows: The states $|\psi_1\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{\imath\phi_1}|1\rangle$ and $|\psi_2\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}e^{\imath(\phi_1+\pi)}|1\rangle$ represent a pair of orthogonal states on the equatorial circle of the Bloch sphere.

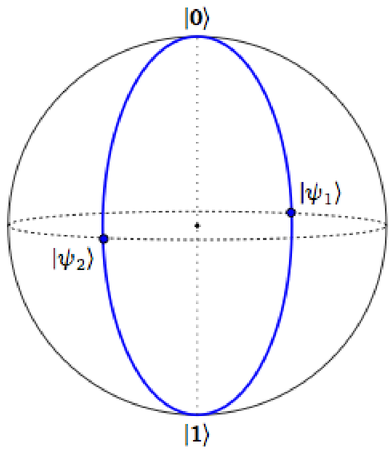

Figure 3.6: The blue circle depicts all the states whose coherence ($l_1 - norm$) the cloner can clone, given the transformation defined in 3.7

Then, any arbitrary state $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$ can be written as $\cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{\imath\phi_2}|1\rangle$ in $\{|0\rangle, |1\rangle\}$ basis. Then, $C_l(|\psi\rangle) = |\sin\theta|$ in $\{|0\rangle, |1\rangle\}$ basis. As $\alpha = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}e^{\imath(\phi_2-\phi_1)})$ and $\beta = \frac{1}{\sqrt{2}}(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}e^{\imath(\phi_2-\phi_1)})$, the final coherence which is given by $C_l(\psi^{final}) = ||\alpha|^2 - |\beta|^2| = |2|\alpha|^2 - 1|$ becomes $|\sin\theta\cos(\phi_2 - \phi_1)|$.

We see that the only solution where initial coherence is equal to final coherence is when $\phi_1 = \phi_2$. Therefore, for all values of θ the final coherence $C_l(\psi^{final}) = C_l(\psi)$ if $\phi_1 = \phi_2$. Which means that the cloner defined in equation 3.7 perfectly clones coherence for all the states on the great circle passing through $|\psi_1\rangle$, $|\psi_2\rangle$, $|0\rangle$, $|1\rangle$ as shown in the Fig. 3.6.

3.4.2 Example of a cloner that can clone coherence of states of same coherence value

Let U_{pp} be the set of unitaries that transforms pure input state and blank state to two pure output states for all input states.

Theorem 9. *There exists no $U \in U_{pp}$ that clone coherence perfectly for states of di erent coherence value.*

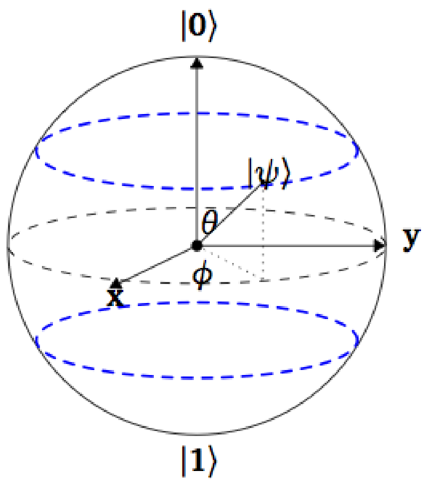

Figure 3.7: Cloner cloning states of same coherence ($l_1 - norm$) value

Proof. Let there be a U $\in U_{pp}$ that follows the transformation rules below.

$$\begin{aligned} |\psi_1\rangle_A|0\rangle_B &\longrightarrow |\psi_1{'}\rangle_A|\psi_1{''}\rangle_B, \\ |\psi_2\rangle_A|0\rangle_B &\longrightarrow |\psi_2{'}\rangle_A|\psi_2{''}\rangle_B. \end{aligned} \tag{3.8}$$

Here $\langle\psi_2|\psi_1\rangle = 0$ and $C_l(\psi_1) = C_l(\psi_1^{'}) = C_l(\psi_1^{''})$ and $C_l(\psi_2) = C_l(\psi_2^{'}) = C_l(\psi_2^{''}) = c$. Either, $\langle\psi_2{'}|\psi_1{'}\rangle = 0$ or $\langle\psi_2{''}|\psi_1{''}\rangle = 0$ because unitaries preserve inner product. Without loss of generality let us assume $\langle\psi_2{'}|\psi_1{'}\rangle = 0$.

Let $|\psi\rangle = \alpha|\psi_1\rangle + \beta|\psi_2\rangle$ be an arbitrary state on which we would like to apply the unitary U. After transformation, the subsystem A will be $\rho_A^{final} = |\alpha|^2|\psi_1{'}\rangle\langle\psi_1{'}| + |\beta|^2|\psi_2{'}\rangle\langle\psi_2{'}| + \langle\psi_2{''}|\psi_1{''}\rangle(\alpha\beta^*|\psi_1{'}\rangle\langle\psi_2{'}| + \alpha^*\beta|\psi_2{'}\rangle\langle\psi_1{'}|)$ and $\rho_B^{final} = |\alpha|^2|\psi_1{''}\rangle\langle\psi_1{''}| + |\beta|^2|\psi_2{''}\rangle\langle\psi_2{''}|$.

If both ρ_A^{final} and ρ_B^{final} are both pure states for any value of α and β then $|\psi_1^{''}\rangle = |\psi_2^{''}\rangle$.

Then, $\rho_A^{final} = (\alpha|\psi_1{'}\rangle + \beta|\psi_2{'}\rangle)(\alpha^*\langle\psi_1{'}| + \beta^*\langle\psi_2{'}|)$ and $\rho_B^{final} = |\psi_1{''}\rangle\langle\psi_1{''}|$ and . Clearly, we can see that the subsystem B, $\rho_B^{final} = |\psi_1{''}\rangle\langle\psi_1{''}|$ is independent of α and β and the final coherence of subsystem B is always c. Hence, we can prove that there exists no U $in\ U_{pp}$ that can clone coherence perfectly for states of different coherence value. □

Note 2: So the U $in\ U_{pp}$ cloners only clone coherence perfectly for states of same coherence value as shown in the Fig. 3.7. One such example of U will be, $U = U_\phi \otimes U_\theta$ where, $U_{\phi'} = \begin{vmatrix} 1 & 0 \\ 0 & \mathbf{e^{-i\phi'}} \end{vmatrix}$

and $U_\theta = \begin{vmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{vmatrix}$, acts on the state $|\psi\rangle$ and the blank state $|0\rangle$ respectively. Here $c = |\sin\theta|$.

3.5 Conclusion

To summarize, it is proved that no-cloning for quantum states does not lead to the no-cloning of coherence. In fact, a stronger statement is proved, that is, the no-cloning of quantum coherence implies the no-cloning of quantum state. Since coherence captures the wave aspect of quantum particles this chapter shows that quantum wave cannot be cloned, where as classical wave can be cloned. In particular, for the input state whose coherence is greater than the coherence of the known states, coherence cloning is not possible. Interestingly, it also showed that the universal cloner does not exist even in the case where there is no ancillary inputs.

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